Supersymmetry of Scattering Amplitudes and Green Functions in Perturbation Theory

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Cerro de la Estrella

Aquí los antiguos recibían al fuego
Aquí el fuego creaba al mundo
Al mediodía las piedras se abren como frutos
El agua abre los párpados
La luz resbala por la piel del día
Gota inmensa donde el tiempo se refleja y se sacia [...]

Octavio Paz, *"Libertad Bajo Palabra"*

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Chapter 1

Introduction and Motivation

Although it is a theory of the utmost accuracy and success, the Standard Model (SM) of elementary particle physics cannot describe Nature up to arbitrarily high energy scales and therefore is not the last answer on our way in uncovering Nature's secrets. Today we look upon the SM as merely an effective field theory which is described by a local, causal quantum field theory up to an energy scale yet unknown, but assumed to lie at about 10^{15} GeV. Though all experimental data available today are in perfect agreement with the description of Nature by the Standard Model, there are some loose ends in the framework of the SM from which we mention just one, the so called *naturalness* or *hierarchy* problem. If the breaking of the electroweak gauge symmetry is provided by an elementary scalar getting a vacuum expectation value, the mass of that scalar, the Higgs boson, should be of the order of the electroweak breaking scale. Typically, the radiative corrections to the mass square of a scalar are proportional to the square of the energy scale at which its quantum field theory is embedded in a more fundamental theory, candidates for which being the Planck scale, a GUT or a string scale of the order given above or higher. This is not the case for fermions which receive only logarithmic corrections. An immense fine tuning for the bare mass of the Higgs scalar at the scale of the more fundamental theory is therefore necessary to cancel the quadratic contributions from the renormalization group flow. If we did not have these cancellations, the "natural" mass square of the Higgs scalar at the electroweak breaking scale would be of the order of the square of the high scale; this is called the naturalness problem. The hierarchy problem means the sheer existence of the vast differences between the two energy scales.

A possible solution of the naturalness problem serves as the strongest motivation for supersymmetry. Supersymmetry is a symmetry which interchanges bosons and fermions and could therefore naturally explain the existence of light scalars. In the supersymmetric limit each fermion loop contributing to the quadratically divergent Higgs self-energy is accompanied by a scalar loop with the opposite sign. Furthermore the coupling constants are required to be equal by supersymmetry, hence the quadratic divergence cancels out and only the logarithmic survives. As a second motivation we may mention gauge coupling unification which is compatible with current data only in supersymmetric extensions of the Standard Model but not in the SM itself. Hence, in spite of technicolour models – theories where the Higgs is a composite object – and models with extra dimensions (whether "large" or not) as competitors, supersymmetric extensions of the SM are the most widely accepted of the hypothesized models beyond the Standard Model.

After the first supersymmetric models had been established in the early 1970s

[?], phenomenology started and supersymmetric extensions of the SM have been constructed, e.g. see the reviews given in [?], [?]. The simplest of these extensions is called the *Minimal Supersymmetric Standard Model* (MSSM), where the predicate "minimal" stands for minimal field content: Each SM field is embedded into a superfield where the SM fermions are accompanied by scalars, the gauge bosons by fermions, called gauginos, and the Higgs bosons also by fermionic superpartners. Moreover, the constraint of being supersymmetric forces the existence of at least two Higgs superfields, one with hypercharge $+1$ and one with hypercharge -1, to give mass to the up- as well as down-type fermions; the appearance of two Higgs doublets is necessary also to avoid anomalies.

Therefore the prediction of supersymmetry is the existence of superpartners for all yet known SM particles. Since they are constrained by SUSY to have the same masses as the SM particles but have not been observed yet, supersymmetry has to be broken. Until today the mechanism of supersymmetry breaking is unknown, so we parameterize our ignorance by the most general explicit breaking of supersymmetry, the so called *soft breaking terms*. They are motivated by the fact that SUSY has to be broken by a whatsoever mechanism at a high scale, producing these explicit breaking terms by the renormalization group evolution of all relevant operators compatible with all symmetries. Though SUSY is a very simple concept and an enormously powerful symmetry, in addition to the huge number of particles, these soft breaking terms make the MSSM tremendously complicated as all particles which are by their quantum numbers allowed to mix really do mix. Also the pure number of free parameters in the MSSM becomes one order of magnitude higher as in the SM, namely 124 [?], or even, in a more general version, 178 [?], [?].

Another issue is the incredible number of vertices considering all Feynman rules of the MSSM (cf. tables **??**, **??**, **??**, **??**) and the sometimes very complex structure of the coupling constants, [?], [?], and [?]. There are some simplifying assumptions for the structure of the coefficients of the soft breaking terms (e.g. *flavour alignment* or *universality*) which are motivated by supergravity embeddings of the MSSM, but need not be fulfilled. One can steer a middle course as a compromise for the model: as general as possible, but as simple as necessary. We choose coefficients which are diagonal in generation space (actually, the generation mixings must be very small not to contradict the experimental thresholds for violation of the separate lepton numbers L_e, L_μ, L_τ) but the diagonal elements need not be equal in contrast to the prejudice given by the universality constraint. The number of vertices in tables **??**, **??**, **??** and **??** has been estimated under this assumption, but even as this is not the most complex of the "minimal" MSSMs, it has a discouraging number of more than four thousand vertices.

Today's generation of running and planned colliders (Tevatron, LHC, and TESLA) will bring the decision which way Nature has chosen for electroweak symmetry breaking (cf. e.g. [?]). But even if a Higgs boson is detected at one of the world's huge colliders in the next years, it will not be easy to determine whether it is a "standard", a minimal supersymmetric, a next to minimal supersymmetric one [?], [?], [?], or something else. For this, extensive knowledge about the alternatives to the SM must be available, and besides the ubiquitous radiative corrections (within the SM, the MSSM and other models), it is indispensable to calculate tree level processes with up to eight particles in the final state, as in highly energetic processes ($10^2 - 10^3$ GeV for the colliders above) the final states are very complex. (The interest in eight final particles comes from the desire to study $WW \to WW$ scattering, the inclusion of the $WWWW$-vertex in eight-fermion production processes, production of $t\bar{t}$-pairs and their decays as well as the production of superpartners and SUSY cascade decays.) Of course, such calculations with $10^4 - 10^8$ participating Feynman diagrams have to be done automatically by matrix element generators like *O'Mega* [?]. Alternative models to the SM have therefore to be incorporated into such matrix element generators as the SM was. The goal for the next years will be to compare possibly found experimental deviations from the SM predictions with the theoretical results from alternative models like the MSSM.

As it soon becomes clear, the work is not done by simply writing a model file for the MSSM to incorporate it in an matrix element generator like *O'Mega*. Since the complexity of the model grows immensely from the SM to the MSSM (compare tables **??**-**??** with tables **??**-**??**) it is inevitable to check the consistency of such models like the MSSM. This is necessary for making sure that all parameters (masses, coupling constants, widths, etc.) are compatible with each other, to debug computer programs (model files, numerical function library, etc.), and not to forget, to have the numerical stability under control. Symmetry principles which have always been strong concepts in physical theories provide such tests for consistency checks here. The MSSM like the SM has its $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry as a powerful tool for those checks; what is often used is the independence of all physical results from the gauge parameter ξ in general R_ξ gauges. Our aim is to make use of the Ward, or better, the Slavnov-Taylor identities of the gauge symmetry [?], [?], [?], [?]. Both kinds of identities originate from the quantum generalization of the symmetry principle of the classical field theory, the first expressing current conservation and being only valuable in the case of global symmetries, the latter stemming from the BRST symmetry left over after gauge fixing.

In supersymmetric field theories we can, of course, use supersymmetry as the underlying symmetry, and there, as long as we are not concerned with local supersymmetry (supergravity), we are able to employ Ward identities. As we will see for supersymmetric gauge theories it is indispensable – even at tree level – to use the Slavnov-Taylor identities. The stringency of the consistency checks is also a drawback: the relations mentioned as vehicles for those tests are quite complicated and involve a number of

Process	# *O'Mega* fusions	# Propagators	# Diagrams
$e^+e^- \to \tilde\chi^0_1\tilde\chi^0_2$	24	8	8
$e^+e^- \to \tilde e^+_1\tilde e^-_1$	27	9	9
$e^+e^- \to \tilde u_1\tilde u^*_1\tilde u_1\tilde u^*_1$	346	41	660
$e^+e^- \to e^+e^-\tilde\chi^0_1\tilde\chi^0_2$	610	60	1,552
$e^+e^- \to \tilde\chi^0_1\tilde\chi^0_2\tilde\chi^0_3\tilde\chi^0_4$	782	66	2,208
$e^+e^- \to \tilde e^+_1\tilde e^-_1\tilde u_1\tilde u^*_1\tilde u_1\tilde u^*_1$	4,002	153	141,486
$e^+e^- \to e^+e^-\mu^+\mu^-\tilde\chi^0_1\tilde\chi^0_2$	4,389	172	239,518
$e^+e^- \to e^+e^-\tilde\chi^0_1\tilde\chi^0_2\tilde\chi^0_3\tilde\chi^0_4$	11,870	280	1,056,810
$e^+e^- \to \tilde\chi^0_1\tilde\chi^0_1\tilde\chi^0_2\tilde\chi^0_2\tilde\chi^0_3\tilde\chi^0_4$	17,075	322	2,191,845
$e^+e^- \to e^+e^-\mu^+\mu^- u\bar u\tilde\chi^0_1\tilde\chi^0_2$	23,272	434	50,285,616
$e^+e^- \to \tilde\chi^0_1\tilde\chi^0_1\tilde\chi^0_2\tilde\chi^0_2\tilde\chi^0_3\tilde\chi^0_3\tilde\chi^0_4\tilde\chi^0_4$	273,950	1,370	470,267,024

Table 1.1: *Juxtaposition of the number of Feynman diagrams and of* O'Mega *fusions for some MSSM processes at a linear collider. By fusions we mean the fundamental calculational steps for constructing the amplitudes in* O'Mega.

sophisticated techniques. As a first and fundamental step, extensive knowledge about how Ward- and Slavnov-Taylor identities for supersymmetric (gauge) field theories work *analytically* (in perturbation theory) has to be gained to use such identities for numerical checks. This will be the concern for the major part of this thesis; first of all, the investigation of the applicability (on-shell and/or off-shell, what kind of method for which model) of the several kinds of methods to be presented here, furthermore – and even more important – to understand the way the cancellations happen in these identities. The latter point is inevitable in deciding which expressions to use in numerical checks: expressions adjusted to the technical nature of cancellations are likely to be numerically more stable than those which are not. A third and last issue then is to transfer these analytical expressions to the matrix element generator and perform numerical checks. Since it is not possible to produce reliable theoretical predictions for future experiments without having powerful consistency checks at hand, and since such consistency checks cannot be under (numerical) control without a deeper understanding of how they work analytically, the original intention of this work has changed: from a purely phenomenological issue at the beginning – to implement realistic supersymmetric models as alternatives to the Standard Model into the matrix element generator *O'Mega* – to a more theoretical one – to develop stringent tests as consistency checks for these models and to understand their fine points in detail. We hope to have convinced the reader that the latter is the *sine qua non* for the first. Thus the main part of this thesis is concerned with analytical perturbative calculations of three different kinds of identities within several models, to our knowledge never been done before. Let us briefly summarize the content of this thesis.

1.1 Structure and Content

After a short introduction to supersymmetry transformations, the main text is divided into four parts, the first showing a method to gain on-shell Ward identities for supersymmetric field theories originally invented in the late 1970s by Grisaru, Pendleton and van Nieuwenhuizen but as far as we know this method has never been used diagrammatically. We investigate that kind of Supersymmetric Ward Identities (SWI) for the Wess-Zumino model and a more complex toy model to uncover some new effects. As this formalism relies on the annihilation of the vacuum by the supercharge, it does not work for spontaneously broken supersymmetry. We provide an example within the framework of the O'Raifeartaigh model.

The second part is concerned with SWI constructed from Green functions with one current insertion and contracted with the momentum brought into the Green function by the current. At tree level these identities are fulfilled on-shell and off-shell. For the latter the SWI are more complicated due to the contributions of several "contact terms" and provide more stringent tests than the on-shell identities. Examples are calculated for the Wess-Zumino model, the toy model from part one and for the O'Raifeartaigh model, as the supersymmetric current is still conserved for spontaneously broken SUSY. It will be shown that this method does not work for supersymmetric gauge theories. The explanation of this phenomenon then blends over to the next part.

There we introduce the BRST formalism for supersymmetric theories where supersymmetry as a global symmetry is quantized with the help of constant ghosts, [?], [?]. In order not to cloud the intricacies by a huge amount of fields and diagrams, we construct the simplest possible supersymmetric Abelian toy model. We summarize the BRST transformations with inclusion of supersymmetry and translations and show several examples of supersymmetric Slavnov-Taylor identities in that toy model and also in supersymmetric QCD.

In the last part we discuss the problems concerned with the implementation of supersymmetric models and the consistency checks mentioned above. Connected with supersymmetric field theories is the appearence of Majorana fermions – real fermions – which are their own antiparticles. The solution of how to let the matrix element generator evaluate the signs coming from Fermi statistics without expanding the Feynman diagrams is presented based on ideas in [**?**]. Furthermore it is presented there how one- and two-point vertices arising together with the BRST formalism can be handled within *O'Mega*, though their topologies are not compatible with the way the amplitudes are built by *O'Mega*. It is demonstrated that Slavnov-Taylor identities for gauge symmetries and supersymmetry can be done within the same framework. Finally we will give an outlook of what remains to be done in that field, possible generalizations and improvements.

Chapter 2

SUSY Transformations

2.1 Classical transformations

First of all, we want to summarize the supersymmetry transformations of classical fields; as a general reference for the basics of supersymmetry we mention the book of Julius Wess and Jonathan Bagger, *Supersymmetry and Supergravity* [?]. By contraction with a fermionic (i.e. Grassmann odd) spinor transformation parameter we make the supercharges bosonic

$$Q(\xi) \equiv \xi Q + \bar{\xi}\bar{Q} \tag{2.1}$$

The component fields of a chiral multiplet, the scalar field ϕ, the Weyl-spinor field ψ and the scalar auxiliary field F with dimension two undergo the following transformations generated by the supercharge $Q(\xi)$ (cf. the appendix as well)

$$\begin{aligned} \delta_\xi \phi &= \sqrt{2}\xi\psi \\ \delta_\xi \psi &= -\mathrm{i}\sqrt{2}\sigma^\mu \bar{\xi}\partial_\mu \phi + \sqrt{2}\xi F \\ \delta_\xi F &= -\mathrm{i}\sqrt{2}\bar{\xi}\bar{\sigma}^\mu \partial_\mu \psi \end{aligned} \tag{2.2}$$

Compared to the book of Wess/Bagger the relative signs in the last two transformations have their origin in the different convention for the metric used by Wess/Bagger. This causes differences in the definition of the 4-vector of the Pauli matrices.

Because $Q(\xi)$ is real (Hermitean as a generator for quantum fields), the transformation properties of a field imply the properties of the complex conjugated field. One simply has to define:

$$(\delta_\xi \Psi)^* = \delta_\xi \Psi^* \quad , \tag{2.3}$$

This is the natural choice for a real generator. The relation will still be fulfilled in the quantized calculus.

Better suited for our aim – application of SUSY transformations in a phenomenological particle physics context – will be a formulation of the transformation rules with bispinors. Therefore we reformulate the transformations given above in this formalism. We also split the lowest and the highest components of the superfields into their scalar and pseudoscalar parts, called “chiral”. This will prove useful later.

$$\phi = \frac{1}{\sqrt{2}}\left(A + \mathrm{i}B\right), \qquad F = \frac{1}{\sqrt{2}}\left(\mathcal{F} - \mathrm{i}\mathcal{G}\right). \tag{2.4}$$

The resulting transformations are:

$$\boxed{\begin{aligned}&\delta_\xi A = \overline{\xi}\psi,\\ &\delta_\xi B = \mathrm{i}\overline{\xi}\gamma^5\psi,\\ &\delta_\xi \psi = -\mathrm{i}\partial\!\!\!/\left(A + \mathrm{i}\gamma^5 B\right)\xi + \left(\mathcal{F} + \mathrm{i}\gamma^5\mathcal{G}\right)\xi,\\ &\delta_\xi \mathcal{F} = -\mathrm{i}\,\overline{\xi}\partial\!\!\!/\psi,\\ &\delta_\xi \mathcal{G} = -\overline{\xi}\partial\!\!\!/\gamma^5\psi.\end{aligned}} \tag{2.5}$$

In this list all spinors are understood as bispinors. For the translation of the "fundamental" component fields to the "chiral" fields we refer to section 2.3.

2.2 SUSY transformations in Hilbert space

The following discussion should prevent the confusion with factors i and signs when talking about SUSY transformations on the classical level and in the context of quantum field theory. Classically we review the results of the last section:

$$\begin{aligned}\delta_\xi\phi \equiv \left(\xi Q + \bar{\xi}\bar{Q}\right), \qquad \left(\xi Q + \bar{\xi}\bar{Q}\right)^* = \xi Q + \bar{\xi}\bar{Q}\\ \delta_\xi\phi^* = \left(\delta_\xi\phi\right)^* = \left(\xi Q + \bar{\xi}\bar{Q}\right)\phi^* \quad ,\end{aligned} \tag{2.6}$$

wherein ϕ could be a field of any geometrical character and any Grassmann parity.

In the quantum theory the transformation is represented by a unitary operator, which is created by exponentiation of the supercharge – now a Hermitean generator – multiplied with i:

$$[\mathrm{i}Q(\xi), \phi] = \delta_\xi\phi \tag{2.7}$$

Again ϕ is a field (operator) of arbitrary geometrical character and Grassmann parity. Moreover, $\delta_\xi\phi$ is the transformation of the classical fields incorporated into Hilbert space, i.e. the classical term, in which the fields have been replaced by operators acting in the Hilbert space. For the Hermitean adjoint one finds:

$$\begin{aligned}[\mathrm{i}Q(\xi), \phi]^\dagger &= \left[\mathrm{i}Q(\xi), \phi^\dagger\right]\\ &= \left(\delta_\xi\phi\right)^\dagger\\ &\Longrightarrow \boxed{\left[\mathrm{i}Q(\xi), \phi^\dagger\right] = \left(\delta_\xi\phi\right)^\dagger = \delta_\xi\phi^\dagger}\end{aligned} \tag{2.8}$$

There is no subtlety in dealing with fermionic fields here because the rule for reversing the order of Grassmann odd parameters classically is translated to the rule for reversing the order of field operators when Hermitean adjoined – no matter whether they are fermionic or bosonic. But one still has to take into account that Grassmann odd classical parameters like ξ and fermionic field operators have to be reversed in order when Hermitean adjoined.

Finally there is a simple rule for the embedding of the classical transformations into the quantum theory: Replace left multiplication with $Q(\xi)$ by application of the commutator with $\mathrm{i}Q(\xi)$.

2.3 General problems with auxiliary fields in supersymmetric field theories

As we will see, there is a possibility to implement SUSY Ward identities for theories with exact supersymmetry and an S-matrix invariant under SUSY transformations, by examining the transformation properties of the creation and annihilation operators of *in* and *out* states. For the extraction of the relations between amplitudes provided by supersymmetry, (in this ansatz) asymptotic fields (cf., for example, Kugo, [?]) have to be taken into account. The only important parts of the asymptotic fields are the one-particle poles, so we only have to keep those terms in the equations of motion of the auxiliary fields F and D which stem from the bilinear parts of the superpotential.

For example in the Wess-Zumino model we have:

$$\begin{aligned} F &= -m\phi^* - \frac{1}{2}\lambda(\phi^*)^2 \\ &= -\frac{m}{\sqrt{2}}(A - \mathrm{i}B) - \frac{\lambda}{4}(A - \mathrm{i}B)^2 \\ &\stackrel{!}{=} \frac{1}{\sqrt{2}}(\mathcal{F} - \mathrm{i}\mathcal{G}) \end{aligned} \tag{2.9}$$

Out of this we obtain the equations of motion for the auxiliary fields:

$$\boxed{\begin{aligned} \mathcal{F} &= -mA - \frac{\lambda}{2\sqrt{2}}\left(A^2 - B^2\right) \\ \mathcal{G} &= -mB - \frac{\lambda}{\sqrt{2}}AB \end{aligned}} \tag{2.10}$$

Off-shell there is no distinction possible between fields and auxiliary fields. The auxiliary fields are necessary to preserve the lemma stating that the number of bosonic and fermionic degrees of freedom has to be equal. For physical processes (with fields on the mass shell) one has to insert the equations of motion for the auxiliary fields. For the derivation of the S-matrix via the LSZ reduction formula all one-particle poles have to be accounted for. This implies further that in the equations of motion only the one-particle poles have to be kept. In the MSSM these poles exclusively appear in the mass terms (soft SUSY breaking terms) and the bilinear Higgs term, the latter also generating masses.

2.4 SUSY transformations of quantum fields

Finally, we are able to write down the SUSY transformations in Hilbert space for the chiral superfield:

$$\boxed{\begin{aligned} [\mathrm{i}Q(\xi), A] &= \overline{\xi}\psi, \\ [\mathrm{i}Q(\xi), B] &= \mathrm{i}\overline{\xi}\gamma^5\psi, \\ [\mathrm{i}Q(\xi), \psi] &= -\mathrm{i}\partial\!\!\!/\left(A + \mathrm{i}\gamma^5 B\right)\xi + \left(\mathcal{F} + \mathrm{i}\gamma^5\mathcal{G}\right)\xi, \\ [\mathrm{i}Q(\xi), \mathcal{F}] &= -\mathrm{i}\overline{\xi}\partial\!\!\!/\psi, \\ [\mathrm{i}Q(\xi), \mathcal{G}] &= -\overline{\xi}\partial\!\!\!/\gamma^5\psi \end{aligned}} \tag{2.11}$$

Taking into account only the one-particle poles, e.g. in the Wess-Zumino model, yields:

$$\boxed{\begin{aligned} &[\mathrm{i}Q(\xi), A] = \overline{\xi}\psi, \\ &[\mathrm{i}Q(\xi), B] = \mathrm{i}\overline{\xi}\gamma^5\psi, \\ &[\mathrm{i}Q(\xi), \psi] = -\left(\mathrm{i}\partial\!\!\!/ + m\right)\left(A + \mathrm{i}\gamma^5 B\right)\xi \end{aligned}} \tag{2.12}$$

Part I

SUSY Ward identities for asymptotic states

Chapter 3

SUSY Ward Identities [SWI] for asymptotic fields

3.1 Consequences of SUSY for S-matrix elements

In supersymmetric field theories supersymmetry is a symmetry of the theory, meaning that the S-operator commutes with the supercharges: $[Q, S] = 0$. Later on we will see that in supersymmetric gauge theories the gauge fixing required for quantization breaks supersymmetry, with the result that the supercharge no longer commutes with the S-operator on the complete Hilbert space but only with the S-operator on the cohomology of the supercharge [**?**]. The S-operator maps the Hilbert space basis of asymptotic *in* states onto the one of the asymptotic *out* states. Therefore we immediately conclude that the *in* and *out* creation and annihilation operators have the same algebra, i.e. commutation relations with the supercharge Q. Remember that we are dealing at the moment with exact supersymmetry, so the vacuum is invariant under SUSY transformations and must be annihilated by the supercharge:

$$Q\,|0\rangle = 0. \tag{3.1}$$

At this point we mention some common grounds and some differences of supersymmetry and BRST symmetry. Both have in common that they are fermionic generators of global symmetries of the theory (we do not treat supergravity and local supersymmetry here) so there are some similarities between them. BRST transformations leave many more states of Hilbert space invariant (namely all physical states) than supersymmetry under which only the vacuum (and perhaps soliton solutions) are invariant. So for constructing relations between amplitudes of different processes we are (in case of supersymmetry) left with on-shell relations between S-matrix amplitudes whereas in BRST identities different off-shell Green functions can be compared. Later on we will bring SUSY and BRST together and derive the most general identities for supersymmetric gauge theories.

For the derivation of SWIs the following relation is the basic ingredient to start with:

$$\boxed{\begin{aligned} 0 &= \left\langle 0 \middle| \left[Q, a_1^{\text{out}} \dots a_n^{\text{out}} a_1^{\dagger\,\text{in}} \dots a_m^{\dagger\,\text{in}} \right] \middle| 0 \right\rangle \\ &= \sum_i \langle 0 | a_1^{\text{out}} \dots [Q, a_i^{\text{out}}] \dots | 0 \rangle + \sum_j \left\langle 0 \middle| a_1^{\text{out}} \dots \left[Q, a_j^{\dagger\,\text{in}} \right] \dots \middle| 0 \right\rangle \end{aligned}} \tag{3.2}$$

It follows, of course, from the invariance of the vacuum under SUSY transformations. So starting with a string of creation operators differing in spin by half a unit from the spin of the annihilation operators we get a sum of amplitudes for different processes where all incoming and outgoing particles are SUSY transformed successively. The creation and annihilation operators needed in the SWI of that kind have to be extracted from the field operators. An explanation for the way this is done will be given in the next section.

3.2 Projecting out creation and annihilation operators

In this section we only summarize the inverse Fourier transformations by which the creation and annihilation operators of excitations of a scalar or fermionic quantum field can be projected out with, following these prescriptions:

$$\boxed{\begin{aligned} a(k) &= \mathrm{i}\int d^3\vec{x}\; e^{\mathrm{i}kx}\, \overleftrightarrow{\partial}_t\, \phi(x) \\ b(k,\sigma) &= \int d^3\vec{x}\;\, \overline{u}(k,\sigma)\,\gamma^0\psi(x)e^{\mathrm{i}kx} \\ d^\dagger(k,\sigma) &= \int d^3\vec{x}\;\, \overline{v}(k,\sigma)\,\gamma^0\psi(x)e^{-\mathrm{i}kx} \end{aligned}} \tag{3.3}$$

In the first line we made use of the famous abbreviation:

$$\left(a\, \overleftrightarrow{\partial}_\mu\, b\right) \equiv a(\partial_\mu b) - (\partial_\mu a)b.$$

In the case of Majorana spinor fields, which are important in supersymmetric field theories, the last two equations are identical. The verification of (3.3) can be found in appendix **??**.

3.3 Transformations of creation and annihilation operators

As was discussed in the first section of this chapter for the derivation of the SWIs we need the SUSY transformation properties of the creation and annihilation operators. To derive them we go back to the so called "chiral" fields, ϕ and ϕ^*, which are now called ϕ_- and ϕ_+. We write down their definitions again:

$$\boxed{\phi_\pm \equiv \frac{1}{\sqrt{2}}\Big(A \mp \mathrm{i}B\Big)} \tag{3.4}$$

At this point, there is a difference in the choice of sign compared to the work of Grisaru, Pendleton and van Nieuwenhuizen [**?**].

Now we are – by the use of the SUSY transformations of the quantum fields and projecting the creation and annihilation operators out of the field operators – able to get the SUSY transformations of the ladder operators. First we discuss the transformations of creation and annihilation operators of the "chiral" scalar fields $\phi_\pm$, for which the notation $a^{(\dagger)}(k,\sigma), \sigma \equiv \pm$ is:

$$\begin{aligned}
[Q(\xi), a(k,\sigma)] &= \mathrm{i} \int d^3\vec{x}\; e^{\mathrm{i}kx} \overleftrightarrow{\partial}_0 \, [Q(\xi), \phi_\sigma(x)] \\
&= -\frac{1}{\sqrt{2}} \int d^3\vec{x}\; e^{\mathrm{i}kx} \overleftrightarrow{\partial}_0 \left(\overline{\xi}(1+\sigma\gamma^5)\psi \right) \\
&= \frac{\mathrm{i}}{\sqrt{2}} \overline{\xi}(1+\sigma\gamma^5) \sum_\tau b(k,\tau) u(k,\tau)
\end{aligned} \tag{3.5}$$

We find the transformation law

$$\boxed{[Q(\xi), a(k,\sigma)] = \frac{\mathrm{i}}{\sqrt{2}} \overline{\xi}(1+\sigma\gamma^5) \sum_\tau b(k,\tau) u(k,\tau)} \tag{3.6}$$

Consider a massless theory, where the spinors $u(k,\tau)$ und $v(k,\tau)$ are eigenstates of the matrix γ^5. We end up with the concise result:

$$[Q(\xi), a(k,\sigma)] = \sqrt{2}\mathrm{i}\, \overline{\xi} u(k,\sigma)\, b(k,\sigma) \tag{3.7}$$

Now we derive the transformation properties for the fermionic annihilation operators:

$$\begin{aligned}
[Q(\xi), b(k,\sigma)] &= \int d^3\vec{x}\; \overline{u}(k,\sigma)\gamma^0 \, [Q(\xi), \psi(x)] \, e^{\mathrm{i}kx} \\
&= -\mathrm{i}\overline{u}(k,\sigma) \left(a_A(k) + \mathrm{i}\gamma^5 a_B(k) \right) \xi,
\end{aligned} \tag{3.8}$$

where we have used the spinor $\overline{u}$'s equation of motion:

$$\overline{u}(p,\sigma)\,(\not{p} - m) = 0. \tag{3.9}$$

When using the chiral fields instead of the scalar and pseudoscalar ones, it follows:

$$\boxed{[Q(\xi), b(k,\sigma)] = -\frac{\mathrm{i}}{\sqrt{2}} \sum_\tau \left(\overline{u}(k,\sigma) \left(1 - \tau\gamma^5\right) \xi \right) a(k,\tau)} \tag{3.10}$$

In the massless case the bispinor is again an eigenstate of the chiral projectors, so we find:

$$[Q(\xi), b(k,\sigma)] = -\sqrt{2}\mathrm{i}\, \overline{u}(k,\sigma)\xi a(k,\sigma). \tag{3.11}$$

We will derive the latter result in a more general context following the discussion of Grisaru and Pendleton [**?**] in section 3.5.

3.4 Anticommutativity, Grassmann numbers and Generators

There is a subtlety which may easily be overlooked, but without it, it is not possible to derive the SUSY transformations of the asymptotic creation operators.

For the quantization of field theories including fermions, Grassmann fields are being used, i.e. spinor fields whose components are Grassmann odd. This is necessary to fulfill the demands of the fermions having Fermi-Dirac statistics. Consider SUSY transformations which contain Grassmann odd constant spinors (as ξ above). Those parameters must *anti*commute with the Fermi fields. Consequently, spinor products normally being skew become symmetric between Fermi fields or between a Fermi field and such a Grassmann odd parameter (There are two signs when interchanging the two spinors in the product, one which causes the skewness of the product, namely the contraction direction of the spinor indices, but also another one from anticommuting the Grassmann numbers (cf. the appendix and [?])). In quantizing such a theory, the anticommutativity must be maintained when going from the classical Fermi fields to the field operators. Because – with the exception of the creation and annihilation operators (about which one could be tempted to assume that they only are responsible for the anticommutativity of fermions on Hilbert space) – there are only commuting terms in the field operators, we have to deduce that the creation and annihilation operators for fermions remain Grassmann odd with respect to "classical" Grassmann numbers. This means

$$\{\xi, b(k,\sigma)\} = \left\{\xi, b^\dagger(k,\sigma)\right\} = \{\xi, d(k,\sigma)\} = \left\{\xi, d^\dagger(k,\sigma)\right\} = 0, \tag{3.12}$$

which has noteworthy technical consequences.

What happens after taking the Hermitean adjoint of an equation like (3.6)? The left hand side yields:

$$\Big([Q(\xi), a(k,\sigma)]\Big)^\dagger = -\left[Q(\xi), a^\dagger(k,\sigma)\right] \tag{3.13}$$

Again we used the Hermiticity of $Q(\xi)$:

$$Q(\xi)^\dagger = \left(\xi Q + \bar{\xi}\bar{Q}\right)^\dagger = \bar{\xi}\bar{Q} + \xi Q = Q(\xi) \tag{3.14}$$

On the right hand side of (3.6) it has been taken into account that a Hermitean adjoint for operators includes complex conjugation of ordinary numbers and Grassmann numbers. The order of Grassmann numbers has to be reversed in complex conjugation:

$$(g_1 g_2 \dots g_n)^* = g_n^* \dots g_2^* g_1^* \qquad g_i \ \text{Grassmann odd} \tag{3.15}$$

One therefore gets:

$$\begin{aligned}\left(\frac{\mathrm{i}}{\sqrt{2}}\overline{\xi}\left(1+\sigma\gamma^5\right)\sum_\tau u(k,\tau)b(k,\tau)\right)^\dagger &= -\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau b^\dagger(k,\tau)u^\dagger(k,\tau)\left(1+\sigma\gamma^5\right)\gamma^0\xi \\ &= -\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau b^\dagger(k,\tau)\overline{u}(k,\tau)\left(1-\sigma\gamma^5\right)\xi \\ &= +\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau \overline{u}(k,\tau)\left(1-\sigma\gamma^5\right)\xi\, b^\dagger(k,\tau)\end{aligned} \tag{3.16}$$

In the last line we used (3.12). This finally produces the relation:

$$\boxed{\left[Q(\xi), a^\dagger(k,\sigma)\right] = -\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau \overline{u}(k,\tau)\left(1-\sigma\gamma^5\right)\xi\, b^\dagger(k,\tau)} \quad . \tag{3.17}$$

Altogether there are three signs: One due to the Hermitean adjoint of the commutator, one by complex conjugation of the explicit factor i and a third one due to the anticommutativity of Fermi field operators and Grassmann numbers.

Another important difficulty about signs, related to the anticommutativity of Fermi field operators and Grassmann numbers, will be discussed in chapter 5.

3.5 General derivation of the transformations

When translating the identities of that kind introduced in the first section of this chapter into the graphical language of Feynman diagrams, we discover several subtleties concerning signs (a trade mark of supersymmetry), which seem to be confusing at the first sight. We discuss these specialties using an example with two incoming and two outgoing particles. Here we have two *in* creation operators and two *out* annihilation operators. With the abbreviation $c_\sigma(k_i)$ instead of $c(k_i,\sigma)$ for $c \equiv a, b$ this SWI reads:

$$\begin{aligned}
0 = &\left\langle 0\middle|\left[Q(\xi), a_-^{\text{out}}(k_3)b_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\right]\middle|0\right\rangle \\
= &\left\langle 0\middle|\left[Q(\xi), a_-^{\text{out}}(k_3)\right] b_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+\left\langle 0\middle|a_-^{\text{out}}(k_3)\left[Q(\xi), b_+^{\text{out}}(k_4)\right] a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+\left\langle 0\middle|a_-^{\text{out}}(k_3)b_+^{\text{out}}(k_4)\left[Q(\xi), a_-^{\dagger\,\text{in}}(k_1)\right] a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+\left\langle 0\middle|a_-^{\text{out}}(k_3)b_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)\left[Q(\xi), a_-^{\dagger\,\text{in}}(k_2)\right]\middle|0\right\rangle
\end{aligned} \tag{3.18}$$

With the help of the relations (3.6), (3.10) and (3.17) this can be transformed into:

$$\begin{aligned}
0 = &\frac{\mathrm{i}}{\sqrt{2}}\sum_\sigma \left\langle 0\middle|\left(\overline{\xi}\mathcal{P}_L u(k_3,\sigma)\right) b_\sigma^{\text{out}}(k_3)b_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&-\frac{\mathrm{i}}{\sqrt{2}}\left\langle 0\middle|a_-^{\text{out}}(k_3)\left(\overline{u}(k_4,+)\mathcal{P}_L\xi\right) a_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&-\frac{\mathrm{i}}{\sqrt{2}}\left\langle 0\middle|a_-^{\text{out}}(k_3)\left(\overline{u}(k_4,+)\mathcal{P}_R\xi\right) a_-^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&-\frac{\mathrm{i}}{\sqrt{2}}\sum_\sigma\left\langle 0\middle|a_-^{\text{out}}(k_3)b_+^{\text{out}}(k_4)\left(\overline{u}(k_1,\sigma)\mathcal{P}_R\xi\right) b_\sigma^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&-\frac{\mathrm{i}}{\sqrt{2}}\sum_\sigma\left\langle 0\middle|a_-^{\text{out}}(k_3)b_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)\left(\overline{u}(k_2,\sigma)\mathcal{P}_R\xi\right) b_\sigma^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle
\end{aligned} \tag{3.19}$$

The sum in (3.10) has been split up so that there are five terms now. To separate the spinor bilinears produced by the SUSY transformations from the S-matrix elements, we bring all these factors to the utmost left. Be aware of picking up a sign in the last two lines by anticommuting the Grassmann odd spinor bilinear and the fermionic annihilator. One ends up with

$$\begin{aligned}
0 = &\frac{\mathrm{i}}{\sqrt{2}}\sum_\sigma\left(\overline{\xi}\mathcal{P}_L u(k_3,\sigma)\right)\left\langle 0\middle|b_\sigma^{\text{out}}(k_3)b_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&-\frac{\mathrm{i}}{\sqrt{2}}\left(\overline{u}(k_4,+)\mathcal{P}_L\xi\right)\left\langle 0\middle|a_-^{\text{out}}(k_3)a_+^{\text{out}}(k_4)a_-^{\dagger\,\text{in}}(k_1)a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mathrm{i}}{\sqrt{2}} \left(\overline{u}(k_4,+)\mathcal{P}_R\xi\right) \left\langle 0 \middle| a_-^{\mathrm{out}}(k_3) a_-^{\mathrm{out}}(k_4) a_-^{\dagger\,\mathrm{in}}(k_1) a_-^{\dagger\,\mathrm{in}}(k_2) \middle| 0 \right\rangle \\
&+ \frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left(\overline{u}(k_1,\sigma)\mathcal{P}_R\xi\right) \left\langle 0 \middle| a_-^{\mathrm{out}}(k_3) b_+^{\mathrm{out}}(k_4) b_\sigma^{\dagger\,\mathrm{in}}(k_1) a_-^{\dagger\,\mathrm{in}}(k_2) \middle| 0 \right\rangle \\
&+ \frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left(\overline{u}(k_2,\sigma)\mathcal{P}_R\xi\right) \left\langle 0 \middle| a_-^{\mathrm{out}}(k_3) b_+^{\mathrm{out}}(k_4) a_-^{\dagger\,\mathrm{in}}(k_1) b_\sigma^{\dagger\,\mathrm{in}}(k_2) \middle| 0 \right\rangle
\end{aligned}
\tag{3.20}
$$

There is yet another source for producing signs, but it can only arise in the context of Dirac fermions – i.e. charged fermions. Anticommutation of fermionic annihilators and/or creators due to the Wick theorem is the origin of these additional sign factors; we will go into the details in chapter 5, which deals with a model in which Dirac fermions appear.

Now we want to revisit part of a general derivation of the SWIs in the formalism originally written down by M.T. Grisaru and H.N. Pendleton used to derive helicity selection rules in gravitino–graviton scattering [?]. Because the supercharges commute with the momentum operator and change the particles' spin by half a unit, we can derive the following relations for the *in* annihilators of particles with spin j and chirality σ in a supersymmetric theory:

$$
\begin{aligned}
[Q(\xi), a_j(k,\sigma)] &= \Delta_j(\xi,k,\sigma) \cdot a_{j-\frac{1}{2}}(k,\sigma), \\
\left[Q(\xi), a_{j-\frac{1}{2}}(k,\sigma)\right] &= \Delta_{j-\frac{1}{2}}(\xi,k,\sigma) \cdot a_j(k,\sigma).
\end{aligned}
\tag{3.21}
$$

The momentum operator has the form:

$$
P^\mu = \sum_\sigma \int d^3\vec{p}\, p^\mu \left(a_j^\dagger(p,\sigma) a_j(p,\sigma) + a_{j-\frac{1}{2}}^\dagger(p,\sigma) a_{j-\frac{1}{2}}(p,\sigma) \right). \tag{3.22}
$$

From the fact that the supercharge and the momentum operator commute, an equation for the two unknown functions Δ_j, $\Delta_{j-\frac{1}{2}}$ on the right hand side can be deduced

$$
\begin{aligned}
[Q(\xi), P^\mu] = &\sum_\sigma \int d^3\vec{p}\, p^\mu \Big(a_j^\dagger(p,\sigma)\, [Q(\xi), a_j(p,\sigma)] + \left[Q(\xi), a_j^\dagger(p,\sigma)\right] a_j(p,\sigma) \\
&\quad + a_{j-\frac{1}{2}}^\dagger(p,\sigma) \left[Q(\xi), a_{j-\frac{1}{2}}(p,\sigma)\right] + \left[Q(\xi), a_{j-\frac{1}{2}}^\dagger(p,\sigma)\right] a_{j-\frac{1}{2}}(p,\sigma) \Big) \\
= &\sum_\sigma \int d^3\vec{p}\, p^\mu \Big(a_j^\dagger(p,\sigma) a_{j-\frac{1}{2}}(p,\sigma) \left(\Delta_j(\xi,p,\sigma) - \Delta_{j-\frac{1}{2}}^*(\xi,p,\sigma) \right) \\
&\quad + a_{j-\frac{1}{2}}^\dagger(p,\sigma) a_j(p,\sigma) \left(\Delta_{j-\frac{1}{2}}(\xi,p,\sigma) - \Delta_j^*(\xi,p,\sigma) \right) \Big) \\
\stackrel{!}{=} &\ 0 \\
\Longrightarrow &\ \Delta_{j-\frac{1}{2}}(\xi,p,\sigma) \ = \ \Delta_j^*(\xi,p,\sigma)
\end{aligned}
\tag{3.23}
$$

Defining $\Delta_j \equiv \Delta$ (3.21) reads

$$
\boxed{
\begin{aligned}
[Q(\xi), a_j(k,\sigma)] &= \Delta(\xi,k,\sigma) \cdot a_{j-\frac{1}{2}}(k,\sigma), \\
\left[Q(\xi), a_{j-\frac{1}{2}}(k,\sigma)\right] &= \Delta^*(\xi,k,\sigma) \cdot a_j(k,\sigma)
\end{aligned}
}\ ,
\tag{3.24}
$$

to be compared with (3.7) and (3.11).

More relations can be gained from the Jacobi identity:

$$\left[\left[Q(\xi), Q(\zeta)\right], a_j(k,\sigma)\right] + \left[\left[Q(\zeta), a_j(k,\sigma)\right], Q(\xi)\right] + \left[\left[a_j(k,\sigma), Q(\xi)\right], Q(\zeta)\right] = 0 \quad (3.25)$$

This implies the equation:

$$\Delta(\zeta, k, \sigma) \cdot \Delta^*(\xi, k, \sigma) - \Delta(\xi, k, \sigma) \cdot \Delta^*(\zeta, k, \sigma) = 2\bar{\xi}\not{k}\zeta \quad . \quad (3.26)$$

As is shown in [**?**], the explicit form of these functions can be found in the context of special models. In the last section we derived them directly by projecting out the annihilators from the field operators. In a general model this procedure can become arbitrarily complicated, especially if one has a nondiagonal metric on the space of states or if unphysical modes are involved.

Chapter 4

The Wess-Zumino Model

We want to test the SUSY Ward identities of the kind derived in the last chapter for the Wess-Zumino (WZ) model. This is the simplest supersymmetric field theoretic model with just one superfield but the most general renormalizable superpotential. For details about the model, the particle content and the Feynman rules see appendix **??**.

4.1 SWI for the WZ model

We can use the formula (3.2) derived in the last chapter to check SWI in the WZ model. The starting point – similar to the derivation of the Slavnov-Taylor identities – is a string of field operators with half integer spin, which only by application of the symmetry generator (here the supercharge), becomes a physically possible (in particular non-vanishing) amplitude. First, we have to translate the formulae from the previous chapter to the physical fields of the WZ model – by this we mean the real and imaginary part of the complex scalar field ϕ or the scalar and pseudoscalar part, respectively.

To get the transformation properties of annihilators and creators of the real part A of the complex scalar field ϕ one has to set the term proportional to γ^5 in equation (3.6) equal to zero and to multiply the result by $\sqrt{2}$. For the imaginary part B one has to set the term proportional to unity equal to zero, to set σ equal to one and multiply the result by a factor $\sqrt{2}\mathrm{i}$. This results in:

$$[Q(\xi), a_A(k)] = \mathrm{i}\sum_\sigma \overline{\xi} u(k,\sigma) b(k,\sigma) \tag{4.1}$$

$$[Q(\xi), a_B(k)] = -\sum_\sigma \overline{\xi}\gamma^5 u(k,\sigma) b(k,\sigma) \quad . \tag{4.2}$$

For the transformation law of the fermion annihilator it suffices to use (3.8),

$$[Q(\xi), b(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k) + \mathrm{i}\gamma^5 a_B(k)\Big)\xi \quad . \tag{4.3}$$

As an example, we take a transformation of a product of an *in* creation operator for one A and one B field, and *out* annihilators for an A field and a Majorana fermion of positive helicity. We therefore write relation (3.2) in the form

$$\begin{aligned} 0 \stackrel{!}{=} & \left\langle 0 \middle| \left[Q(\xi), a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \right] \middle| 0 \right\rangle \\ = & \quad \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) \left[Q(\xi), a_B^{\dagger\,\text{in}}(k_2) \right] \middle| 0 \right\rangle \\ & + \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) \left[Q(\xi), a_A^{\dagger\,\text{in}}(k_1) \right] a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\ & + \left\langle 0 \middle| a_A^{\text{out}}(k_3) \left[Q(\xi), b^{\text{out}}(k_4,+) \right] a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\ & + \left\langle 0 \middle| \left[Q(\xi), a_A^{\text{out}}(k_3) \right] b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \end{aligned} \tag{4.4}$$

This seems to relate the amplitudes of four different physical processes. But as the transformation of a fermionic annihilator produces a linear combination of annihilators for the scalar and pseudoscalar fields, A and B, respectively, we get indeed five different processes (here we adopt the convention that processes only differing by the helicity of a fermion are counted as *one* process).

$$\begin{aligned} 0 \stackrel{!}{=} & -\sum_\sigma \overline{u}(k_2,\sigma)\gamma^5\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) b^{\dagger\,\text{in}}(k_2,\sigma) \middle| 0 \right\rangle \\ & + \mathrm{i} \sum_\sigma \overline{u}(k_1,\sigma)\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) b^{\dagger\,\text{in}}(k_1,\sigma) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\ & - \mathrm{i}\, \overline{u}(k_4,+)\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) a_A^{\text{out}}(k_4) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\ & + \overline{u}(k_4,+)\gamma^5\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) a_B^{\text{out}}(k_4) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\ & + \mathrm{i} \sum_\sigma \overline{\xi} u(k_3,\sigma) \left\langle 0 \middle| b^{\text{out}}(k_3,\sigma) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \end{aligned} \tag{4.5}$$

Note the double sign arising in the last two lines – as explained in section 3.1 – coming from a relative sign between the transformation properties of a creation and an annihilation operator and one from equation (3.17). With the help of the relation for S-matrix elements and amplitudes, which e.g. can be read off from [**?**], p. 105,

$$\begin{aligned} & \langle q_1 \ldots q_n | S | p_1 \ldots p_m \rangle_{\text{conn.}} = \\ & \qquad \mathrm{i}\mathcal{M}(p_1,\ldots,p_m \longrightarrow q_1,\ldots,q_n)\, (2\pi)^4 \delta^4 \left(\sum_{i=1}^m p_i - \sum_{j=1}^n q_j \right) \quad , \end{aligned} \tag{4.6}$$

equation (4.5) can immediately be transferred into Feynman diagrams (omitting the overall factor i and also the delta function for global momentum conservation):

$$\begin{aligned} 0 \stackrel{!}{=} & -\sum_\sigma \overline{u}(k_2,\sigma)\gamma^5\xi \cdot \mathcal{M}(A(k_1)\Psi(k_2,\sigma) \longrightarrow A(k_3)\Psi(k_4,+)) \\ & + \mathrm{i} \sum_\sigma \overline{u}(k_1,\sigma)\xi \cdot \mathcal{M}(\Psi(k_1,\sigma)B(k_2) \longrightarrow A(k_3)\Psi(k_4,+)) \\ & - \mathrm{i}\overline{u}(k_4,+)\xi \cdot \mathcal{M}(A(k_1)B(k_2) \longrightarrow A(k_3)A(k_4)) \\ & + \overline{u}(k_4,+)\gamma^5\xi \cdot \mathcal{M}(A(k_1)B(k_2) \longrightarrow A(k_3)B(k_4)) \\ & + \mathrm{i} \sum_\sigma \overline{\xi} u(k_3,\sigma) \cdot \mathcal{M}(A(k_1)B(k_2) \longrightarrow \Psi(k_3,\sigma)\Psi(k_4,+)) \quad . \end{aligned} \tag{4.7}$$

Diagrammatically we can write down the following expression:

$$
\begin{aligned}
0 = & -\sum_\sigma \overline{u}(k_2,\sigma)\gamma^5\xi \cdot \left\{ \qquad + \qquad + \qquad \right\} \\
& + \mathrm{i} \sum_\sigma \overline{u}(k_1,\sigma)\xi \cdot \left\{ \qquad + \qquad + \qquad \right\} + 0 \\
& + \overline{u}(k_4,+)\gamma^5\xi \cdot \left\{ \qquad + \qquad + \qquad + \qquad \right\} \\
& + \mathrm{i} \sum_\sigma \overline{\xi} u(k_3,\sigma) \cdot \left\{ \qquad + \qquad + \qquad \right\}
\end{aligned}
\tag{4.8}
$$

For the calculation of the amplitudes it is useful to introduce the Mandelstam variables,

$$s = (k_1 + k_2)^2 = (k_3 + k_4)^2 \ , \tag{4.9}$$

$$t = (k_3 - k_1)^2 = (k_4 - k_2)^2 \ , \tag{4.10}$$

$$u = (k_4 - k_1)^2 = (k_3 - k_2)^2 \ . \tag{4.11}$$

The explicit analytical expressions for diagrams in which only scalar (or pseudoscalar) particles are involved are easily found and work in the same manner as in ϕ^4 theory or the Standard Model. For the diagrams with Majorana fermions the Feynman rules for general fermions worked out by Denner et al. [**?**] are needed.

The terms in braces yield the following analytical expressions, in the first line of (4.8)

$$-\frac{\mathrm{i}\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{3m}{t-m^2} + \frac{\not k_1 + \not k_2 + m}{s-m^2} + \frac{\not k_2 - \not k_3 + m}{u-m^2}\right) u(k_2,\sigma) \quad , \tag{4.12}$$

in the second line

$$-\frac{\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{m\gamma^5}{u-m^2} + \frac{(\not k_1 + \not k_2 + m)\gamma^5}{s-m^2} + \frac{\gamma^5(\not k_1 - \not k_3 + m)}{t-m^2}\right) u(k_1,\sigma) \quad . \tag{4.13}$$

The diagrams in the third line add up to

$$-\frac{\mathrm{i}\lambda^2}{2}\left(\frac{3m^2}{t-m^2} + \frac{m^2}{s-m^2} + \frac{m^2}{u-m^2} + 1\right) \quad , \tag{4.14}$$

and finally in the last line of (4.8):

$$-\frac{\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{m\gamma^5}{s-m^2}+\frac{\gamma^5(\not k_4-\not k_2+m)}{t-m^2}+\frac{(\not k_4-\not k_1+m)\gamma^5}{u-m^2}\right)v(k_3,\sigma)\quad . \tag{4.15}$$

It proves to be more convenient for further simplification – remember that we still have to multiply the prefactors from equation (4.8) – to modify the analytical expression for the diagrams in the last line. To apply the spin summation formula

$$\sum_\sigma u(p,\sigma)\overline{u}(p,\sigma)=\not p+m \tag{4.16}$$

we reverse the calculational direction of the Majorana fermion line for the last process. How this works is explained in detail in [?]. The result looks like

$$+\frac{\lambda^2}{2}\,\overline{u}(k_3,\sigma)\left(\frac{m\gamma^5}{s-m^2}+\frac{(\not k_3-\not k_1+m)\gamma^5}{t-m^2}+\frac{\gamma^5(\not k_3-\not k_2+m)}{u-m^2}\right)v(k_4,+)\quad , \tag{4.17}$$

with the change in sign coming from the antisymmetry of the charge conjugation matrix. There are no additional signs from the vertices because all couplings are scalar or pseudoscalar (cf. again [?]). It is important to keep track of the momenta's signs in the fermion propagators.

Equation (4.8) now has the form:

$$\begin{aligned}
0 \stackrel{!}{=}&\\
&\frac{\mathrm{i}\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{3m}{t-m^2}+\frac{\not k_1+\not k_2+m}{s-m^2}+\frac{\not k_2-\not k_3+m}{u-m^2}\right)(\not k_2+m)\gamma^5\xi\\
&-\frac{\mathrm{i}\lambda^2}{2}\overline{u}(k_4,+)\left(\frac{m\gamma^5}{u-m^2}+\frac{(\not k_1+\not k_2+m)\gamma^5}{s-m^2}+\frac{\gamma^5(\not k_1-\not k_3+m)}{t-m^2}\right)(\not k_1+m)\xi\\
&-\frac{\mathrm{i}\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{3m^2}{t-m^2}+\frac{m^2}{s-m^2}+\frac{m^2}{u-m^2}+1\right)\gamma^5\xi\\
&+\frac{\mathrm{i}\lambda^2}{2}\overline{\xi}(\not k_3+m)\left(\frac{m\gamma^5}{s-m^2}+\frac{(\not k_3-\not k_1+m)\gamma^5}{t-m^2}+\frac{\gamma^5(\not k_3-\not k_2+m)}{u-m^2}\right)v(k_4,+)
\end{aligned} \tag{4.18}$$

We divide everything by the common factor $\frac{\mathrm{i}\lambda^2}{2}$. To achieve the same structure for all four contributions we reverse the fermion line in the last process a second time to arrive at

$$\begin{aligned}
0 \stackrel{!}{=}&\\
&\overline{u}(k_4,+)\left(\frac{3m}{t-m^2}+\frac{\not k_1+\not k_2+m}{s-m^2}+\frac{\not k_2-\not k_3+m}{u-m^2}\right)(\not k_2+m)\gamma^5\xi\\
&-\overline{u}(k_4,+)\left(\frac{m\gamma^5}{u-m^2}+\frac{(\not k_1+\not k_2+m)\gamma^5}{s-m^2}+\frac{\gamma^5(\not k_1-\not k_3+m)}{t-m^2}\right)(\not k_1+m)\xi\\
&-\overline{u}(k_4,+)\left(\frac{3m^2}{t-m^2}+\frac{m^2}{s-m^2}+\frac{m^2}{u-m^2}+1\right)\gamma^5\xi\\
&-\overline{u}(k_4,+)\left(\frac{m\gamma^5}{s-m^2}+\frac{\gamma^5(\not k_4-\not k_2+m)}{t-m^2}+\frac{(\not k_4-\not k_1+m)\gamma^5}{u-m^2}\right)(-\not k_3+m)\xi
\end{aligned} \tag{4.19}$$

The terms proportional to m^2 in the first and third row cancel and we are left with:

$$\begin{aligned}
0 \stackrel{!}{=} \overline{u}(k_4,+)\Bigg[& \frac{3m\not{k}_2}{t-m^2} + \frac{(\not{k}_1+\not{k}_2)(\not{k}_2+m)+m\not{k}_2}{s-m^2} + \frac{(\not{k}_2-\not{k}_3)(\not{k}_2+m)+m\not{k}_2}{u-m^2} \\
& + \frac{m(\not{k}_1-m)}{u-m^2} + \frac{(\not{k}_1+\not{k}_2+m)(\not{k}_1-m)}{s-m^2} \\
& - \frac{(\not{k}_1-\not{k}_3-m)(\not{k}_1-m)}{t-m^2} - 1 - \frac{m(\not{k}_3+m)}{s-m^2} \\
& + \frac{(\not{k}_4-\not{k}_2-m)(\not{k}_3+m)}{t-m^2} - \frac{(\not{k}_4-\not{k}_1+m)(\not{k}_3+m)}{u-m^2}\Bigg]\gamma^5\xi
\end{aligned} \tag{4.20}$$

Considering the terms proportional to $(t-m^2)^{-1}$ and applying the Dirac equation,

$$\overline{u}(k_4,+)\,(\not{k}_4-m) = 0 \quad , \tag{4.21}$$

and momentum conservation

$$k_1+k_2 = k_3+k_4 \quad , \tag{4.22}$$

one gets

$$\begin{aligned}
& (t-m^2)^{-1}\Big[3m\not{k}_2 + \not{k}_2(\not{k}_1-m) - \not{k}_2(\not{k}_3+m)\Big] \\
&= (t-m^2)^{-1}\Big[m\not{k}_2 + \not{k}_2(\not{k}_1-\not{k}_3)\Big] \quad = \quad (t-m^2)^{-1}\Big[\not{k}_4\not{k}_2+\not{k}_2\not{k}_4-m^2\Big] \\
&= (t-m^2)^{-1}\Big[2(k_2k_4)-m^2\Big] \quad = \quad (t-m^2)^{-1}(-t+m^2) \\
&= \; -1
\end{aligned} \tag{4.23}$$

The terms proportional to $(s-m^2)^{-1}$ add up to

$$\begin{aligned}
& (s-m^2)^{-1}\Big[\not{k}_1\not{k}_2 + m^2 + m(\not{k}_1+\not{k}_2) + m\not{k}_2 + \not{k}_2\not{k}_1 - m\not{k}_2 - m\not{k}_3 - m^2\Big] \\
&= (s-m^2)^{-1}\Big[\not{k}_1\not{k}_2 + \not{k}_2\not{k}_1 + m(\not{k}_1+\not{k}_2-\not{k}_3)\Big] \\
&= (s-m^2)^{-1}\Big[2(k_1k_2)+m^2\Big] \quad = \quad (s-m^2)^{-1}(s-m^2) \\
&= \; +1,
\end{aligned} \tag{4.24}$$

while the remaining u terms yield:

$$\begin{aligned}
& (u-m^2)^{-1}\Big[m\not{k}_2 + m^2 - m\not{k}_3 - \not{k}_3\not{k}_2 + m\not{k}_2 + m\not{k}_1 - m^2 - \not{k}_2\not{k}_3 - m\not{k}_2\Big] \\
&= (u-m^2)^{-1}\Big[-\not{k}_2\not{k}_3 - \not{k}_3\not{k}_2 + m(\not{k}_1+\not{k}_2-\not{k}_3)\Big] \\
&= (u-m^2)^{-1}\Big[-2(k_2k_3)+m^2\Big] \quad = \quad (u-m^2)^{-1}(u-m^2) \\
&= \; +1 \qquad .
\end{aligned} \tag{4.25}$$

So finally all terms add up to zero and the SWI is fulfilled.

4.2 Jacobi identities for the WZ model

An important possibility to test the consistency of the SWI themselves is to check whether the Jacobi identities for the appearing operators, i.e. the supercharge and the annihilation and creation operators for the particles, are valid.

In the sequel we frequently will use the properties of Grassmann odd bilinears under the exchange of the two spinors. These can e.g. be found in [?] (cf. also appendix ??):

$$\overline{\eta}\Gamma\xi = \begin{cases} +\overline{\xi}\Gamma\eta & \text{für } \Gamma = 1, \gamma^5, \gamma^5\gamma^\mu \\ -\overline{\xi}\Gamma\eta & \text{für } \Gamma = \gamma^\mu, [\gamma^\mu, \gamma^\nu] \end{cases} \tag{4.26}$$

There is no complication in proving the Jacobi identities for the scalar annihilation operators:

$$-\Big[[Q(\xi), Q(\eta)]\,, a_A(k)\Big] \stackrel{!}{=} \Big[[Q(\eta), a_A(k)]\,, Q(\xi)\Big] + \Big[[a_A(k), Q(\xi)]\,, Q(\eta)\Big] \tag{4.27}$$

For the left hand side we have

$$\text{LHS (4.27)} \;=\; -\left[2\overline{\xi}\not{P}\eta, a_A(k)\right] = +2\left(\overline{\xi}\not{k}\eta\right) a_A(k) \quad .$$

The right hand side results in

$$\begin{aligned} \text{RHS (4.27)} \;&=\; -\,\mathrm{i}\sum_\sigma \overline{\eta}u(k,\sigma)\,[Q(\xi), b(k,\sigma)] - (\xi \leftrightarrow \eta) \\ &=\; -\sum_\sigma \overline{\eta}u(k,\sigma)\,\overline{u}(k,\sigma)\Big(a_A(k) + \mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi \leftrightarrow \eta) \\ &=\; -\,\overline{\eta}\,(\not{k} + m)\,\Big(a_A(k) + \mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi \leftrightarrow \eta) \\ &=\; -\,(\overline{\eta}\not{k}\xi)a_A(k) + (\overline{\xi}\not{k}\eta)a_A(k) \;\;=\;\; 2(\overline{\xi}\not{k}\eta)a_A(k) \qquad \surd \end{aligned}$$

The calculation for the annihilator of the pseudoscalar particle B is analogous, the only difference being the appearance of γ^5, which lets the parts containing a_A vanish and those with a_B remain.

$$-\Big[[Q(\xi), Q(\eta)]\,, a_B(k)\Big] = \Big[[Q(\eta), a_B(k)]\,, Q(\xi)\Big] + \Big[[a_B(k), Q(\xi)]\,, Q(\eta)\Big] \tag{4.28}$$

$$\text{LHS (4.28)} \;=\; -\left[2\overline{\xi}\not{P}\eta, a_B(k)\right] = +2\left(\overline{\xi}\not{k}\eta\right) a_B(k)$$

$$\begin{aligned} \text{RHS (4.28)} \;&=\; +\sum_\sigma \overline{\eta}\gamma^5 u(k,\sigma)\,[Q(\xi), b(k,\sigma)] - (\xi \leftrightarrow \eta) \\ &=\; -\,\mathrm{i}\sum_\sigma \overline{\eta}\gamma^5 u(k,\sigma)\,\overline{u}(k,\sigma)\Big(a_A(k) + \mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi \leftrightarrow \eta) \\ &=\; -\,\mathrm{i}\overline{\eta}\gamma^5\,(\not{k} + m)\,\Big(a_A(k) + \mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi \leftrightarrow \eta) \\ &=\; -\,(\overline{\eta}\not{k}\xi)a_B(k) + (\overline{\xi}\not{k}\eta)a_B(k) \;\;=\;\; 2(\overline{\xi}\not{k}\eta)a_B(k) \qquad \surd \end{aligned}$$

A more complicated task is the calculation of the Jacobi identity for the fermion annihilators. We are forced to use the Fierz transformations, the Gordon identity and all other formulae for spinors needed before. First of all the Jacobi identity has, of course, the same form as usual:

$$-\Big[[Q(\xi),Q(\eta)]\,,b(k,\sigma)\Big] \stackrel{!}{=} \Big[[Q(\eta),b(k,\sigma)]\,,Q(\xi)\Big] + \Big[[b(k,\sigma),Q(\xi)]\,,Q(\eta)\Big] \qquad (4.29)$$

For the momentum operator on the left hand side one has to insert only the part of the particle number operators of the fermions, which yields

$$\text{LHS (4.29)} \;=\; -\left[2\overline{\xi}\not{P}\eta, b(k,\sigma)\right] = +2\left(\overline{\xi}\not{k}\eta\right) b(k,\sigma) \quad .$$

The right hand side can be manipulated in the following way:

$$\begin{aligned}
\text{RHS (4.29)} \;&=\; +\mathrm{i}\overline{u}(k,\sigma)\Big([Q(\xi),a_A(k)] + \mathrm{i}\gamma^5\,[Q(\xi),a_B(k)]\Big)\eta - (\xi\leftrightarrow\eta) \\
&=\; -\sum_\tau (\overline{u}(k,\sigma)\eta)(\overline{\xi}u(k,\tau))b(k,\tau) \\
&\quad\; +\sum_\tau (\overline{u}(k,\sigma)\gamma^5\eta)(\overline{\xi}\gamma^5 u(k,\tau))b(k,\tau) - (\xi\leftrightarrow\eta)
\end{aligned}$$

To calculate these products of spinor bilinears we have to use the Fierz identities to be found in appendix **??** as well as e.g. in [**?**]. For arbitrary commuting spinors $\lambda_i, i = 1,\dots,4$ we therefore introduce these abbreviations:

$$\begin{aligned}
s(4,2;3,1) &= (\overline{\lambda}_4\lambda_2)\,(\overline{\lambda}_3\lambda_1) \\
v(4,2;3,1) &= (\overline{\lambda}_4\gamma^\mu\lambda_2)\,(\overline{\lambda}_3\gamma_\mu\lambda_1) \\
t(4,2;3,1) &= \frac{1}{2}(\overline{\lambda}_4\sigma^{\mu\nu}\lambda_2)\,(\overline{\lambda}_3\sigma_{\mu\nu}\lambda_1) \qquad (4.30)\\
a(4,2;3,1) &= (\overline{\lambda}_4\gamma^5\gamma^\mu\lambda_2)\,(\overline{\lambda}_3\gamma_\mu\gamma^5\lambda_1) \\
p(4,2;3,1) &= (\overline{\lambda}_4\gamma^5\lambda_2)\,(\overline{\lambda}_3\gamma^5\lambda_1)
\end{aligned}$$

The scalar and pseudoscalar combinations (take care of the sign which has to be accounted for in case of spinors 2 and 3 being Grassmann odd!) give us the following relations:

$$s(4,2;3,1) = -\frac{1}{4}\Big(s(4,1;3,2)+v(4,1;3,2)+t(4,1;3,2)+a(4,1;3,2)+p(4,1;3,2)\Big) \qquad (4.31)$$

$$p(4,2;3,1) = -\frac{1}{4}\Big(s(4,1;3,2)-v(4,1;3,2)+t(4,1;3,2)-a(4,1;3,2)+p(4,1;3,2)\Big) \qquad (4.32)$$

Due to equation (4.26) the scalar, the pseudoscalar and the pseudovector are symmetric under interchange of the two Grassmann odd spinors, hence after subtracting the "exchange" term $(\xi\leftrightarrow\eta)$ these contributions vanish. The scalar and pseudoscalar combination appear on the right hand side of equation (4.29) with different signs, so the tensorial part of the equation cancels. Only the vector contribution remains four times (scalar/pseudoscalar and a factor two by adding the "exchange" term), so we have

$$\text{RHS (4.29)} \;=\; +\sum_\tau \left(\overline{u}(k,\sigma)\gamma^\mu u(k,\tau)\right)\left(\overline{\xi}\gamma_\mu\eta\right) b(k,\tau) \qquad (4.33)$$

Finally the Gordon identity (cf. e.g. [?], eq. (2.54))

$$\overline{u}(p,\sigma)\gamma^\mu u(p',\tau) = \frac{1}{2m}\overline{u}(p,\sigma)\Big((p+p')^\mu + \mathrm{i}\sigma^{\mu\nu}(p-p')_\nu\Big)u(p',\tau) \tag{4.34}$$

for identical momenta $p = p' \equiv k$ is used, that is why the second term vanishes. With the normalization of the Dirac spinors

$$\overline{u}(k,\sigma)u(k,\tau) = 2m\,\delta_{\sigma\tau} \tag{4.35}$$

the polarization sum over τ collapses and we end up with the desired result

$$\text{RHS (4.29)} \;=\; +2\,\left(\overline{\xi}\not{k}\eta\right) b(k,\sigma)\;. \qquad \surd \tag{4.36}$$

Chapter 5

A toy model

5.1 General remarks

To study the effects stemming from mixings of component fields from different superfields – independent of the difficulty of spontaneous breakdown of supersymmetry as in the O'Raifeartaigh model – we consider another toy model. It consists of two superfields, a mass term and a trilinear coupling. Like for the WZ model we summarize details about the model and the derivation of the Feynman rules in appendix **??**.

5.2 SUSY transformations of Dirac spinors

The main difference between this toy model and the WZ model is the problem of diagonalizing the mass terms which arise by the existence of more than one (at least two as here) superfields. By fusing a left- and a righthanded Weyl spinor from different superfields (not connected through Hermitean adjoint) a Dirac bispinor has been constructed. Moreover there is the problem of "clashing arrows" in Feynman diagrams, i.e. vertices with apparently incompatible directions of the fermion lines. More accurately this means the appearance of two fermions or two antifermions attached to a vertex in such models. This may happen if quadratic terms of superfields, whose fermionic components are combined into Dirac spinors, appear in the trilinear part of the superpotential. Another possibility is within the kinetic terms of the vector superfields in the Lagrangean density of supersymmetric gauge theories if their fermionic components are combined into Dirac fermions together with the Weyl components of chiral matter superfields, as is the case for the charginos in the MSSM.

First of all we want to derive the SUSY transformations of the scalar annihilators, in analogy to the calculations in chapter 3. The mode expansions of the *charged* scalar fields – the scalar component fields of the second superfield – are as follows

$$\begin{aligned}\phi(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2E}\left(a_-(p)e^{-\mathrm{i}px} + a_+^\dagger(p)e^{+\mathrm{i}px}\right)\\ \phi^*(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2E}\left(a_+(p)e^{-\mathrm{i}px} + a_-^\dagger(p)e^{+\mathrm{i}px}\right)\end{aligned} \tag{5.1}$$

Analogously, the projection onto the two different annihilators results in

$$a_-(k) = \mathrm{i}\int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, \phi(x), \qquad a_+(k) = \mathrm{i}\int d^3\vec{x}\ \overleftrightarrow{\partial}_t\, \phi^*(x) \tag{5.2}$$

This enables us to write down the transformation laws of the annihilators.

$$\begin{aligned}[Q(\xi), a_+(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, [Q(\xi), \phi^*(x)] \\ &= -\sqrt{2} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \left(\bar{\xi}\mathcal{P}_R\chi_2(x)\right)\end{aligned}$$

Here and in the sequel χ_1 and χ_2 are the Majorana bispinors which could be built of the fermionic component fields of the first and the second superfield,

$$\chi_1 = \begin{pmatrix} \psi_1 \\ \bar{\psi}_1 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} \psi_2 \\ \bar{\psi}_2 \end{pmatrix}. \tag{5.3}$$

With the definition of the Dirac field (**??**) we are able to express the righthanded Majorana field in terms of the Dirac field:

$$\Psi = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix} = \mathcal{P}_L\chi_1 + \mathcal{P}_R\chi_2 \quad \Longrightarrow \quad \mathcal{P}_R\chi_2 = \mathcal{P}_R\Psi \tag{5.4}$$

Inserting this in the above equation and performing a calculation in the same manner as in chapter 3 one finally gets the relation

$$\boxed{[Q(\xi), a_+(k)] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\bar{\xi}\mathcal{P}_R u(k,\sigma)\right) b(k,\sigma)} \tag{5.5}$$

Trying to proceed analogously for the annihilator $a_-(k)$ reveals a problem,

$$\begin{aligned}[Q(\xi), a_-(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, [Q(\xi), \phi(x)] \\ &= -\sqrt{2} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \left(\bar{\xi}\mathcal{P}_L\chi_2(x)\right) \quad ,\end{aligned}$$

which consists of an impossibility – at first look – to express the lefthanded Majorana field built of the spinor components of the second superfield in terms of the components of the Dirac field. The solution is to pass over to the charge conjugated Dirac field,

$$\Psi^c \equiv \mathcal{C}\overline{\Psi}^T = \begin{pmatrix} \psi_2 \\ \bar{\psi}_1 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{P}_L\chi_2 = \mathcal{P}_L\Psi^c \ , \tag{5.6}$$

with the charge conjugation matrix $\mathcal{C}$. Remembering the mode expansion of the charge conjugated field operator,

$$\Psi^c(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \sum_\sigma \Big(u(p,\sigma)d(p,\sigma)e^{-\mathrm{i}px} + v(p,\sigma)b^\dagger(p,\sigma)e^{\mathrm{i}px}\Big), \tag{5.7}$$

the result for the SUSY transformation of the antifermion annihilator is found:

$$\boxed{[Q(\xi), a_-(k)] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\bar{\xi}\mathcal{P}_L u(k,\sigma)\right) d(k,\sigma)} \tag{5.8}$$

How to project the annihilation operators out of the scalar component fields is well known by now:

$$\begin{aligned} a_A(k) &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, A(x) \\ a_B(k) &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, B(x). \end{aligned} \tag{5.9}$$

The derivation of the transformation laws is at first identical to those of the annihilators $a_+(k)$ and $a_-(k)$:

$$
\begin{aligned}
[Q(\xi), a_A(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \, [Q(\xi), A(x)] \\
&= -\int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \, (\bar{\xi}\chi_1)
\end{aligned}
$$

$$
\begin{aligned}
[Q(\xi), a_B(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \, [Q(\xi), B(x)] \\
&= -\mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \, (\bar{\xi}\gamma^5\chi_1)
\end{aligned}
$$

The difference to the scalar fields of the second superfield is, that now the whole Majorana spinor fields and not only the left- or righthanded parts are present. In consequence, the Dirac spinor field and its charge conjugate both appear in the transformation laws for $a_A(k)$ and $a_B(k)$ according to the expansion

$$
\chi_1 = \begin{pmatrix} \psi_1 \\ \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{\psi}_1 \end{pmatrix} = \mathcal{P}_L \Psi + \mathcal{P}_R \Psi^c \tag{5.10}
$$

After inserting the above we arrive at the final form of the transformation laws for $a_A(k)$ and $a_B(k)$, which yield linear combinations of the Dirac fermion's particle and antiparticle annihilation operators:

$$
\boxed{
\begin{aligned}
[Q(\xi), a_A(k)] &= \mathrm{i} \sum_\sigma \Big(\big(\bar{\xi}\mathcal{P}_L u(k,\sigma)\big)\, b(k,\sigma) + \big(\bar{\xi}\mathcal{P}_R u(k,\sigma)\big)\, d(k,\sigma) \Big) \\
[Q(\xi), a_B(k)] &= \sum_\sigma \Big(\big(\bar{\xi}\mathcal{P}_L u(k,\sigma)\big)\, b(k,\sigma) - \big(\bar{\xi}\mathcal{P}_R u(k,\sigma)\big)\, d(k,\sigma) \Big)
\end{aligned}
}
\tag{5.11}
$$

Remark: If the annihilators $b(k,\sigma)$ and $d(k,\sigma)$ are identical we have a real, i.e. a Majorana fermion and the equations (5.11) are reduced to the relations (4.1) and (4.2). For the chiral scalar fields ϕ and ϕ^* the same is true if we identify b and d and form the linear combinations $(\sqrt{2})^{-1}\,(\phi + \phi^*)$ and $(\mathrm{i}\sqrt{2})^{-1}\,(\phi - \phi^*)$, respectively. Hence the generalization of the Wess-Zumino model for Dirac fermions is consistent.

Deriving the SUSY transformations of the fermionic annihilators is more complicated. We must be aware of the fact that the Dirac bispinor field is composed from the Weyl spinor field ψ_1 as its lefthanded component and from the Weyl spinor field $\bar{\psi}_2$ as its righthanded component. Only these two chiral fields appear (we did not construct a Majorana bispinor field of the component fields $\psi_{1/2}$ and $\bar{\psi}_{1/2}$ from the first chiral superfield or from the second superfield, respectively) which means that here we only have to consider the transformations of the components of the leftchiral superfield $\hat{\Phi}_1$ and the rightchiral superfield $\hat{\Phi}_2^\dagger$ and not of their Hermitean adjoints. Everything is consistent and chirality is conserved. Going back to the roots, the transformation laws are:

$$
\begin{aligned}
[Q(\xi), \mathcal{P}_L\chi_1] = \mathcal{P}_L\,[Q(\xi), \chi_1] &= -\mathrm{i}\mathcal{P}_L(\mathrm{i}\partial\!\!\!/)\,\big(A + \mathrm{i}\gamma^5 B\big)\,\xi + \mathrm{i}\sqrt{2}\,F_1\mathcal{P}_L\xi \\
&\to -\mathrm{i}(\mathrm{i}\partial\!\!\!/)\mathcal{P}_R\,\big(A + \mathrm{i}\gamma^5 B\big)\,\xi - \mathrm{i}m\sqrt{2}\phi^*\mathcal{P}_L\xi \\
&= -\mathrm{i}(\mathrm{i}\partial\!\!\!/)\,(A + \mathrm{i}B)\,\mathcal{P}_R\xi - \mathrm{i}m\sqrt{2}\phi^*\mathcal{P}_L\xi
\end{aligned}
\tag{5.12}
$$

In the second line we inserted the equation of motion for the auxiliary field F_1 and took the one-particle pole for the asymptotic fields of the theory. By the same method one gets for the righthanded fermion field of the second superfield

$$\begin{aligned}[Q(\xi), \mathcal{P}_R \chi_2] = \mathcal{P}_R \left[Q(\xi), \chi_2\right] &= -\,\mathrm{i}\mathcal{P}_R(\mathrm{i}\partial\!\!\!/)\sqrt{2}\,(\mathcal{P}_R\phi + \mathcal{P}_L\phi^*)\,\xi + \mathrm{i}\sqrt{2}\,F_2^*\mathcal{P}_R\xi \\ &\to -\,\mathrm{i}(\mathrm{i}\partial\!\!\!/)\sqrt{2}\mathcal{P}_L\,(\mathcal{P}_R\phi + \mathcal{P}_L\phi^*)\,\xi - \mathrm{i}m\,(A + \mathrm{i}B)\,\mathcal{P}_R\xi \\ &= -\,\mathrm{i}(\mathrm{i}\partial\!\!\!/)\sqrt{2}\phi^*\mathcal{P}_L\xi - \mathrm{i}m\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi \quad .\end{aligned} \tag{5.13}$$

Here, for inserting the poles of the asymptotic fields into the equation of motion for the auxiliary field F_2, it is important to note that for the SUSY transformation of the lefthanded Weyl spinor field the auxiliary field is multiplied with the lefthanded Grassmann spinor ξ, whereas for the transformation of the righthanded Weyl spinor field we have the complex conjugated auxiliary field multiplied by the righthanded Grassmann spinor $\bar{\xi}$ (cf. chapter 1, and [?], [?], [?]).

Combining the two transformation laws (5.12) and (5.13) one reaches

$$\begin{aligned}[Q(\xi), \Psi] &= \mathcal{P}_L\left[Q(\xi), \chi_1\right] + \mathcal{P}_R\left[Q(\xi), \chi_2\right] \\ &= -\,\mathrm{i}(\mathrm{i}\partial\!\!\!/ + m)\,(A + \mathrm{i}B)\,\mathcal{P}_R\xi - \mathrm{i}(\mathrm{i}\partial\!\!\!/ + m)\sqrt{2}\,\phi^*\,\mathcal{P}_L\xi\end{aligned} \tag{5.14}$$

With the help of the equations (3.3) from chapter 3 we are able to deduce the SUSY transformations of the asymptotic annihilation operators (and as a by-product also those for the creation operators). The calculations are analogous to those in (3.8) so that the positive-frequency part (the one with the annihilators) remains.

$$\begin{aligned}[Q(\xi), b(k,\sigma)] &= \int d^3\vec{x}\,\overline{u}(k,\sigma)\gamma^0 e^{\mathrm{i}kx}\,[Q(\xi), \Psi(x)] \\ &= -\,\mathrm{i}\int d^3\vec{x}\,\overline{u}(k,\sigma)\gamma^0 e^{\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/ + m)\,(A + \mathrm{i}B)\,\mathcal{P}_R\xi \\ &\quad -\,\mathrm{i}\int d^3\vec{x}\,\overline{u}(k,\sigma)\gamma^0 e^{\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/ + m)\sqrt{2}\,\phi^*\,\mathcal{P}_L\xi\end{aligned}$$

This implies:

$$\boxed{[Q(\xi), b(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k)\mathcal{P}_R + \mathrm{i}a_B(k)\mathcal{P}_R + \sqrt{2}a_+(k)\mathcal{P}_L\Big)\xi} \tag{5.15}$$

Finally, we reconsider in detail the calculation for the antifermion creator on which originally is projected, wherein we use the notation $k = (E, \vec{k})$ und $\tilde{k} = (E, -\vec{k})$:

$$\begin{aligned}\left[Q(\xi), d^\dagger(k,\sigma)\right] &= \int d^3\vec{x}\,\overline{v}(k,\sigma)\gamma^0 e^{-\mathrm{i}kx}\,[Q(\xi), \Psi(x)] \\ &= -\,\mathrm{i}\int d^3\vec{x}\,\overline{v}(k,\sigma)\gamma^0 e^{-\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/ + m)\,(A + \mathrm{i}B)\,\mathcal{P}_R\xi \\ &\quad -\,\mathrm{i}\int d^3\vec{x}\,\overline{v}(k,\sigma)\gamma^0 e^{-\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/ + m)\sqrt{2}\,\phi^*\,\mathcal{P}_L\xi \\ &= -\,\mathrm{i}\int \frac{d^3\vec{x}\,d^3\vec{p}}{(2\pi)^3 2E}\overline{v}(k,\sigma)\gamma^0\Big((p\!\!\!/ + m)\,(a_A(p) + \mathrm{i}a_B(p))\,e^{-\mathrm{i}(k+p)x} \\ &\qquad - (p\!\!\!/ - m)\,\Big(a_A^\dagger(p) + \mathrm{i}a_B^\dagger(p)\Big)\,e^{\mathrm{i}(p-k)x}\Big)\mathcal{P}_R\xi\end{aligned}$$

$$
\begin{aligned}
& - \mathrm{i}\sqrt{2} \int \frac{d^3\vec{x}\, d^3\vec{p}}{(2\pi)^3 2E} \overline{v}(k,\sigma)\gamma^0 \Big((\not{p} + m)\, a_+(p) e^{-\mathrm{i}(k+p)x} \\
& \qquad\qquad - (\not{p} - m)\, a_-^\dagger(p) e^{\mathrm{i}(k-p)x} \Big) \mathcal{P}_L \xi \\
= & - \frac{\mathrm{i}}{2E} \overline{v}(k,\sigma)\gamma^0 \Big((\tilde{\not{k}} + m) \left(a_A(\tilde{k}) + \mathrm{i} a_B(\tilde{k}) \right) \\
& \qquad\qquad - (\not{k} - m) \left(a_A^\dagger(k) + \mathrm{i} a_B^\dagger(k) \right) \Big) \mathcal{P}_R \xi \\
& - \frac{\sqrt{2}\mathrm{i}}{2E} \overline{v}(k,\sigma)\gamma^0 \Big((\tilde{\not{k}} + m)\, a_+(\tilde{k}) - (\not{k} - m)\, a_-^\dagger(k) \Big) \mathcal{P}_L \xi \\
= & - \frac{\mathrm{i}}{2E} \overline{v}(k,\sigma) \Big((\not{k} + m)\, \gamma^0 \left(a_A(\tilde{k}) + \mathrm{i} a_B(\tilde{k}) \right) \\
& \qquad\qquad + \left(-2E\gamma^0 + \not{k} + m \right) \gamma^0 \left(a_A^\dagger(k) + \mathrm{i} a_B^\dagger(k) \right) \Big) \mathcal{P}_R \xi \\
& - \frac{\sqrt{2}\mathrm{i}}{2E} \overline{v}(k,\sigma) \Big((\not{k} + m)\, \gamma^0 a_+(\tilde{k}) + \left(-2E\gamma^0 + \not{k} + m \right) \gamma^0 a_-^\dagger(k) \Big) \mathcal{P}_L \xi \\
= & + \mathrm{i} \overline{v}(k,\sigma) \Big(a_A^\dagger(k) \mathcal{P}_R + \mathrm{i} a_B^\dagger(k) \mathcal{P}_R + \sqrt{2} a_-^\dagger(k) \mathcal{P}_L \Big) \xi
\end{aligned}
$$

In the last line we used the Dirac equation in the form $\overline{v}(k,\sigma)\,(\not{k} + m) = 0$. Complex conjugation changes this result into

$$[Q(\xi), d(k,\sigma)] = +\mathrm{i}\overline{\xi} \Big(a_A(k)\mathcal{P}_L - \mathrm{i} a_B(k)\mathcal{P}_L + \sqrt{2} a_-(k)\mathcal{P}_R \Big) v(k,\sigma).$$

"Reversing the calculational direction of the fermion line" with respect to the Feynman rules [?] (this way of speaking originates from changing the calculational directions of fermion lines in diagrams and refers to the property of fermion bilinears summarized in appendix ??) gives rise to the final result:

$$\boxed{[Q(\xi), d(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma) \Big(a_A(k)\mathcal{P}_L - \mathrm{i} a_B(k)\mathcal{P}_L + \sqrt{2} a_-(k)\mathcal{P}_R \Big) \xi} \tag{5.16}$$

5.3 A cross-check: Jacobi identities

The Jacobi identities for this toy model are mostly in complete analogy to the Jacobi identities for the WZ model, but there are some fine points which have to be handled carefully. So we show the calculations in detail here.

The Jacobi identity has the standard structure:

$$-\Big[[Q(\xi), Q(\eta)]\,, a_A(k) \Big] = \Big[[Q(\eta), a_A(k)]\,, Q(\xi) \Big] + \Big[[a_A(k), Q(\xi)]\,, Q(\eta) \Big] \tag{5.17}$$

Up to now it is well known how to manipulate the left hand side

$$\text{LHS (5.17)} = +2\left(\overline{\xi}\not{k}\eta \right)\, a_A(k) \tag{5.18}$$

There are more steps to take on the right hand side compared to the case of the WZ model and they are a little bit more complex, too,

$$\text{RHS (5.17)} = \ \mathrm{i} \sum_\sigma \left(\overline{\eta}\mathcal{P}_L u(k,\sigma) \right) [b(k,\sigma), Q(\xi)]$$

$$
\begin{aligned}
&+\mathrm{i}\sum_{\sigma}\left(\overline{\eta}\mathcal{P}_R u(k,\sigma)\right)\left[d(k,\sigma),Q(\xi)\right]-(\xi\leftrightarrow\eta)\\
=\;&-\overline{\eta}\mathcal{P}_L\left(\not k+m\right)\Big(a_A(k)\mathcal{P}_R+\mathrm{i}a_B(k)\mathcal{P}_R+\sqrt{2}\mathcal{P}_L a_+(k)\Big)\xi\\
&-\overline{\eta}\mathcal{P}_R\left(\not k+m\right)\Big(a_A(k)\mathcal{P}_L-\mathrm{i}a_B(k)\mathcal{P}_L+\sqrt{2}\mathcal{P}_R a_-(k)\Big)\xi-(\xi\leftrightarrow\eta)\\
=\;&-\left(\overline{\eta}\mathcal{P}_L\not k\xi\right)a_A(k)-\mathrm{i}\left(\overline{\eta}\mathcal{P}_L\not k\xi\right)a_B(k)-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_L\xi\right)a_+(k)\\
&-\left(\overline{\eta}\mathcal{P}_R\not k\xi\right)a_A(k)+\mathrm{i}\left(\overline{\eta}\mathcal{P}_R\not k\xi\right)a_B(k)-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_R\xi\right)a_+(k)-(\xi\leftrightarrow\eta)\\
=\;&+2\left(\overline{\xi}\not k\eta\right)a_A(k)\quad\surd
\end{aligned}
$$

In the second equation we used the polarization sum for the Dirac spinors $u(k,\sigma)$, in the third equation the anticommutativity of γ^5 with the other gamma matrices and finally, in the fourth equation, we made use of the identity (4.26), which, after subtracting the term $(\xi\leftrightarrow\eta)$, forces the scalar and pseudoscalar parts to vanish so that only the vector contribution with the annihilator $a(k,\sigma)$ remains.

The calculation for the annihilation operator of the pseudoscalar particle, $a_B(k)$, is almost completely analogous.

What about the annihilators of the chiral scalar fields, i.e. the component fields from the second supermultiplet? The difference lies only in the commutator of the supercharge with the annihilator now producing either the fermion or the antifermion annihilator. In particular,

$$-\Big[\left[Q(\xi),Q(\eta)\right],a_+(k)\Big]=\Big[\left[Q(\eta),a_+(k)\right],Q(\xi)\Big]+\Big[\left[a_+(k),Q(\xi)\right],Q(\eta)\Big],\tag{5.19}$$

$$-\Big[\left[Q(\xi),Q(\eta)\right],a_-(k)\Big]=\Big[\left[Q(\eta),a_-(k)\right],Q(\xi)\Big]+\Big[\left[a_-(k),Q(\xi)\right],Q(\eta)\Big].\tag{5.20}$$

The left hand sides look as usual,

$$\text{LHS (5.19)}=+2\left(\overline{\xi}\not k\eta\right)a_+(k),\qquad\text{LHS (5.20)}=+2\left(\overline{\xi}\not k\eta\right)a_-(k)\quad.$$

No problems show up for the right hand sides:

$$
\begin{aligned}
\text{RHS (5.19)}=\;&\mathrm{i}\sum_{\sigma}\left(\overline{\eta}\mathcal{P}_R u(k,\sigma)\right)\left[b(k,\sigma),Q(\xi)\right]-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\overline{\eta}\mathcal{P}_R\left(\not k+m\right)\Big(a_A(k)\mathcal{P}_R+\mathrm{i}a_B(k)\mathcal{P}_R+\sqrt{2}a_+(k)\mathcal{P}_L\Big)\xi-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_R\xi\right)a_A(k)-\sqrt{2}\,\mathrm{i}m\left(\overline{\eta}\mathcal{P}_R\xi\right)a_B(k)\\
&-2\left(\overline{\eta}\mathcal{P}_R\not k\xi\right)a_+(k)-(\xi\leftrightarrow\eta)\\
=\;&+2\left(\overline{\xi}\not k\eta\right)a_+(k)\quad\surd\qquad,
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS (5.20)}=\;&\mathrm{i}\sum_{\sigma}\left(\overline{\eta}\mathcal{P}_L u(k,\sigma)\right)\left[d(k,\sigma),Q(\xi)\right]-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\overline{\eta}\mathcal{P}_L\left(\not k+m\right)\Big(a_A(k)\mathcal{P}_L-\mathrm{i}a_B(k)\mathcal{P}_L+\sqrt{2}a_-(k)\mathcal{P}_R\Big)\xi-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_L\xi\right)a_A(k)+\sqrt{2}\,\mathrm{i}m\left(\overline{\eta}\mathcal{P}_L\xi\right)a_B(k)\\
&-2\left(\overline{\eta}\mathcal{P}_L\not k\xi\right)a_-(k)-(\xi\leftrightarrow\eta)\\
=\;&+2\left(\overline{\xi}\not k\eta\right)a_-(k)\quad\surd\qquad,
\end{aligned}
$$

where the last line again follows from (4.26).

There is nothing new about the Jacobi identities of the fermion annihilation operators, but for the sake of completeness, we list the calculations here, too. Again we have the standard structure

$$-\Big[\left[Q(\xi),Q(\eta)\right],b(k,\sigma)\Big] = \Big[\left[Q(\eta),b(k,\sigma)\right],Q(\xi)\Big] + \Big[\left[b(k,\sigma),Q(\xi)\right],Q(\eta)\Big] \tag{5.21}$$

$$-\Big[\left[Q(\xi),Q(\eta)\right],d(k,\sigma)\Big] = \Big[\left[Q(\eta),d(k,\sigma)\right],Q(\xi)\Big] + \Big[\left[d(k,\sigma),Q(\xi)\right],Q(\eta)\Big] \tag{5.22}$$

The left hand sides are:

$$\text{LHS (5.21)} = +2\left(\overline{\xi}\not{k}\eta\right)b(k,\sigma) \quad ,$$

$$\text{LHS (5.22)} = +2\left(\overline{\xi}\not{k}\eta\right)d(k,\sigma) \quad .$$

For the right hand side we find

$$\begin{aligned}
\text{RHS (5.21)} = & -\mathrm{i}\overline{u}(k,\sigma)\Big(\left[a_A(k),Q(\xi)\right]\mathcal{P}_R + \mathrm{i}\left[a_B(k),Q(\xi)\right]\mathcal{P}_R \\
& \quad + \sqrt{2}\left[a_+(k),Q(\xi)\right]\mathcal{P}_L\Big) - (\xi\leftrightarrow\eta) \\
= & -\sum_\tau\left(\overline{u}(k,\sigma)\mathcal{P}_R\eta\right)\Big(\overline{\xi}\mathcal{P}_L u(k,\tau)b(k,\tau) + \overline{\xi}\mathcal{P}_R u(k,\tau)d(k,\tau)\Big) \\
& -\sum_\tau\left(\overline{u}(k,\sigma)\mathcal{P}_R\eta\right)\Big(\overline{\xi}\mathcal{P}_L u(k,\tau)b(k,\tau) - \overline{\xi}\mathcal{P}_R u(k,\tau)d(k,\tau)\Big) \\
& -2\sum_\tau\left(\overline{u}(k,\sigma)\mathcal{P}_L\eta\right)\left(\overline{\xi}\mathcal{P}_R u(k,\tau)\right)b(k,\tau) \quad - \quad (\xi\leftrightarrow\eta)
\end{aligned}$$

Obviously the contributions of the antifermion annihilators cancel out. In this calculation, by multiplying out the chiral spinor bilinears, one gets the same scalar and pseudoscalar terms as for the Jacobi identity for the fermion annihilator of the WZ model (4.29), so we can use that earlier result.

$$\begin{aligned}
\text{RHS (5.21)} = & -2\sum_\tau\Big(\left(\overline{u}(k,\sigma)\mathcal{P}_R\eta\right)\left(\overline{\xi}\mathcal{P}_L u(k,\tau)\right) \\
& \quad + \left(\overline{u}(k,\sigma)\mathcal{P}_L\eta\right)\left(\overline{\xi}\mathcal{P}_R u(k,\tau)\right)\Big)b(k,\tau) - (\xi\leftrightarrow\eta) \\
= & -\sum_\tau\left(\overline{u}(k,\sigma)\eta\right)\left(\overline{\xi}u(k,\tau)\right)b(k,\tau) \\
& +\sum_\tau\left(\overline{u}(k,\sigma)\gamma^5\eta\right)\left(\overline{\xi}\gamma^5 u(k,\tau)\right)b(k,\tau) - (\xi\leftrightarrow\eta) \\
= & +2\left(\overline{\xi}\not{k}\eta\right)\ b(k,\sigma) \quad \checkmark
\end{aligned}$$

The calculation for $d(k,\sigma)$ is analogous.

5.4 Wick theorem and plenty of signs

Another point of utmost importance appears whenever charged fermions come into play: We have to take care of relative signs between amplitudes belonging to different processes in the same SWI. This is due to the Wick theorem, with the signs stemming from disentangling the contractions of the interaction operators of Yukawa type $\overline{\Psi}\Psi\phi$. To illuminate this further, we want to show an example considering the SWI:

$$0 = \left\langle 0 \middle| \left[Q(\xi), a_A^{\text{out}}(k_3) d^{\text{out}}(k_4,+) a_A^{\text{in}\,\dagger}(k_1) a_A^{\text{in}\,\dagger}(k_2) \right] \middle| 0 \right\rangle \tag{5.23}$$

This produces a relation between the following processes of the diagrammatical form (for the vertices and propagators see appendix **??**):

$$\begin{aligned} 0 = {} & \boxed{(-1)\cdot}\, \mathrm{i} \sum_\sigma \left(\overline{\xi} \mathcal{P}_L u(k_3,\sigma)\right) \cdot \left\{ \qquad + \qquad + \qquad \right\} \\ & - \mathrm{i}\overline{u}(k_4,+)\mathcal{P}_L \xi \cdot \left\{ \qquad + \qquad + \qquad + \qquad \right\} \\ & + \boxed{(-1)\cdot}\, \mathrm{i} \sum_\sigma \left(\overline{u}(k_1,\sigma) \mathcal{P}_L \xi\right) \cdot \left\{ \qquad + \qquad + \qquad \right\} \\ & + \boxed{(-1)\cdot}\, \mathrm{i} \sum_\sigma \left(\overline{u}(k_2,\sigma) \mathcal{P}_L \xi\right) \cdot \left\{ \qquad + \qquad + \qquad \right\} \end{aligned} \tag{5.24}$$

Here we have omitted several processes giving vanishing contributions, $AA \to AB$, $AA \to A\phi^{(*)}$, $AA \to \overline{\Psi}\,\overline{\Psi}$ and $A\Psi \to A\overline{\Psi}$. At first glance, the signs in boxes might seem totally arbitrary, but can be verified by the Wick theorem. Before proving this statement we show that without these signs the SWI would indeed not be valid.

The calculation for the SWI is principally analogous to similar calculations in chapter 4 done within the WZ model. Thus we may omit the details here. No difficulties arise as we can switch directly from analytical Feynman rules to diagrams. We use the polarization sum of Dirac spinors and the change of sign, but not of chirality, when "reversing" a fermion line, [**?**],

$$-\mathrm{i}\left(\overline{u}(k_4,+)\mathcal{P}_L\xi\right) = +\mathrm{i}\left(\overline{\xi}\mathcal{P}_L v(k_4,+)\right) \qquad . \tag{5.25}$$

The first process $A(k_1)A(k_2) \to \Psi(k_3,\sigma)\overline{\Psi}(k_4,+)$ yields, after multiplication with its prefactor and performing the polarization sum,

$$- 2g^2\,\overline{\xi}\mathcal{P}_L\Bigg(\frac{3m\,(\not{k}_3+m)}{s-m^2}+\frac{(\not{k}_3+m)\,(\not{k}_3-\not{k}_2+m)}{t-m^2} + \frac{(\not{k}_3+m)\,(\not{k}_3-\not{k}_1+m)}{u-m^2}\Bigg)v(k_4,+) \qquad . \tag{5.26}$$

For the purely scalar process $A(k_1)A(k_2)\to A(k_3)A(k_4)$ we have to "reverse a fermion line" (i.e. the spinorial prefactor) as mentioned above

$$+6g^2\,\overline{\xi}\mathcal{P}_L\Bigg(\frac{3m^2}{s-m^2}+\frac{3m^2}{t-m^2}+\frac{3m^2}{u-m^2}+1\Bigg)v(k_4,+) \qquad . \tag{5.27}$$

The scattering $A(k_1)\overline{\Psi}(k_2,\sigma)\to A(k_3)\overline{\Psi}(k_4,+)$ of the scalar particle and the antifermion sums up to give the amplitude as follows:

$$- 2g^2\,\overline{\xi}\mathcal{P}_L\Bigg(\frac{-3m\,(\not{k}_2-m)}{u-m^2}+\frac{(\not{k}_2-m)\,(\not{k}_1+\not{k}_2-m)}{s-m^2} + \frac{(\not{k}_2-m)\,(\not{k}_2-\not{k}_3-m)}{t-m^2}\Bigg)v(k_4,+) \tag{5.28}$$

With the help of the substitutions $k_1\leftrightarrow k_2$ and $t\leftrightarrow u$ we get the amplitude for the remaining fourth process $\overline{\Psi}(k_1,\sigma)A(k_2)\to A(k_3)\overline{\Psi}(k_4,+)$.

Summing up the amplitudes of these four processes with the appropriate prefactors gives zero. The calculation is totally identical to the corresponding one done in the WZ model. Now it is obvious that the three added signs are necessary for the SWI to be fulfilled. But where do they come from?

Take a look at the first process as an S-matrix element:

$$\Big\langle 0\Big|b^{\mathrm{out}}(k_3,\sigma)d^{\mathrm{out}}(k_4,+)a_A^{\mathrm{in}\,\dagger}(k_1)a_A^{\mathrm{in}\,\dagger}(k_2)\Big|0\Big\rangle \tag{5.29}$$

When examining the three diagrams in the first line of (5.24), the following expression arises, where we suppress the momentum and spin arguments as well as the *in* and *out* labels,

$$\langle 0|b\;d\;(\overline{\Psi}\Psi A)\,(AAA)a^\dagger a^\dagger|0\rangle = (-1)\cdot\langle 0|b(\overline{\Psi}\Psi d A)\,(AAA)a^\dagger a^\dagger|0\rangle$$

To disentangle the contraction lines we had to anticommute the fermion annihilation operators. We used the conventional notations for contractions

$$\begin{aligned}
A(x)A(y) &= \int\frac{d^4p}{(2\pi)^4}e^{-\mathrm{i}p(x-y)}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}\\
\langle 0|\,d(p,\sigma)\Psi &= v(p,\sigma)\\
A\;a^\dagger\,|0\rangle &= 1\\
\dots\quad&\dots
\end{aligned}$$

Using them, we can correctly convert the Feynman rules into analytical expressions. By means of this anticommutation, a sign emerges. One is easily convinced that the SWI with a fermion in the final state instead of an antifermion does not need this anticommutation. Due to the reversed order of the two fermion annihilation operators, no such sign arises in that case. After a short calculation we find that the two other diagrams contributing to the process considered above pick up signs by the same mechanism, whereas this would not be the case if the two diagrams contained the fermion instead of the antifermion annihilator in the S-matrix element.

The structure of these signs can be understood with the help of [**?**], on top of page 4. From there we can read off the sign of an S-matrix element to be $(-1)^{P+L+V}$, where L is the number of closed fermion loops, P is the parity of the permutation of asymptotic annihilators and creators after having disentangled the fermion lines, and V is the number of incoming and outgoing antifermions. We do only deal with tree level diagrams here, so the number of loops always is zero and no sign is produced by them. The signs stemming from the permutation of the ladder operators are the same as those between the different contributions from s- and t-channel in Bhabha scattering. We had already taken them into account for the WZ model. While we had only Majorana fermions there and could have contracted the field operators in an arbitrary way with the ladder operators for external particles, the fact that we now have to handle Dirac fermions and the sign problem connected with the existence of antifermions discussed in [**?**] is a new topic arising within our toy model. The signs in boxes in (5.24) are due to this effect.

Because we need some additional techniques for calculating an SWI for $(2 \to 2)$-processes, we show a detailed calculation here, starting with three fermions. In that case vertices with "clashing arrows" will appear. This is examplified with

$$0 \stackrel{!}{=} \left\langle 0 \right| \left[Q(\xi), a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) b^{\text{in}\,\dagger}(k_1,+) d^{\text{in}\,\dagger}(k_2,-) \right] \left| 0 \right\rangle \quad . \tag{5.30}$$

For the first process, $\Psi(k_1,+)\overline{\Psi}(k_2,-) \to \overline{\Psi}(k_3,\sigma)\Psi(k_4,+)$, five diagrams contribute,

$$+ \qquad\qquad + \qquad\qquad - \qquad\qquad - \tag{5.31}$$

The relative sign of the third diagram (containing the "clashing arrows") has to be determined carefully from the Wick theorem and depends on the "position of the fermion lines relative to each other". More signs possibly arise here, depending on the calculational directions of the fermion lines as explained in [**?**]; this can happen, if it is necessary to anticommute the two fermion field operators in the interaction terms. Nevertheless this is compensated (cf. again [**?**]) by additional signs produced at the gamma matrices attached to the vertices, giving the same result. For the last two diagrams the relative signs, too, stem from the Wick theorem and can be understood as belonging to exchange diagrams in the same manner as for Bhabha scattering. The positive sign of the third diagram can be seen as belonging to a u-channel, as the u-channel diagram has a relative sign with respect to the t-channel diagrams but not to the s-channel diagrams as in quantum electrodynamics (Of course, without Feynman number violating vertices it is not possible to have s-, t- *and* u-channel diagrams there). But the global sign (which is indispensable for comparison with the other processes contributing to the SWI) is only calculable with the Wick theorem. For more complicated processes it is inevitable to use the Wick theorem to get the correct signs. Fortunately, as will be discussed later, it is possible to do this in a way compatible with the *O'Mega* factorization procedure.

The five diagrams give, together with all signs and after summing over the spin σ:

$$
\begin{aligned}
\text{1. process, (5.31)} \ = 4g^2 \Bigg\{ & \frac{1}{s-m^2}\Big(m\,(\overline{v}(k_2,-)\mathcal{P}_L u(k_1,+))\,(\overline{u}(k_4,+)\mathcal{P}_R\xi) \\
& \qquad - (\overline{v}(k_2,-)\mathcal{P}_R u(k_1,+))\,(\overline{u}(k_4,+)\not{k}_3\mathcal{P}_R\xi)\Big) \\
& + \frac{1}{t-m^2}\Big((\overline{u}(k_4,+)\mathcal{P}_R u(k_1,+))\,(\overline{v}(k_2,-)\not{k}_3\mathcal{P}_R\xi) \\
& \qquad - m\,(\overline{u}(k_4,+)\mathcal{P}_L u(k_1,+))\,(\overline{v}(k_2,-)\mathcal{P}_R\xi)\Big) \\
& - \frac{1}{u-m^2}\,(\overline{u}(k_4,+)\mathcal{P}_R u(k_2,-))\,(\overline{v}(k_1,+)\not{k}_3\mathcal{P}_R\xi)\Bigg\}
\end{aligned}
$$

The second process is decomposed into the two separate parts $\Psi(k_1,+)\overline{\Psi}(k_2,-) \to A(k_3)A(k_4)$,

$$
+ \qquad\qquad + \tag{5.32}
$$

as well as $\Psi(k_1,+)\overline{\Psi}(k_2,-) \to A(k_3)B(k_4)$:

$$
+ \qquad\qquad + \tag{5.33}
$$

It is not difficult to derive the analytical expressions. For (5.32) we get

$$
-2g^2\,\overline{v}(k_2,-)\left(\frac{3m}{s-m^2} + \frac{\not{k}_1 - \not{k}_3 + m}{u-m^2} + \frac{\not{k}_1 - \not{k}_4 + m}{t-m^2}\right)u(k_1,+)\ \ (\overline{u}(k_4,+)\mathcal{P}_R\xi)\,,
$$

and for (5.33):

$$
-2g^2\,\overline{v}(k_2,-)\left(\frac{m}{s-m^2} - \frac{\not{k}_1 - \not{k}_3 - m}{u-m^2} + \frac{\not{k}_1 - \not{k}_4 + m}{t-m^2}\right)\gamma^5 u(k_1,+)\ \ (\overline{u}(k_4,+)\mathcal{P}_R\xi)
$$

SUSY transforming the antifermion in the initial state again gives rise to two different processes, $\Psi(k_1,+)A(k_2) \to A(k_3)\Psi(k_4,+)$,

$$
+ \qquad\qquad + \tag{5.34}
$$

and $\Psi(k_1,+)B(k_2) \to A(k_3)\Psi(k_4,+)$:

$$
+ \qquad\qquad + \tag{5.35}
$$

The corresponding terms are:

$$2g^2\,\overline{u}(k_4,+)\left(\frac{3m}{t-m^2}+\frac{\not{k}_1+\not{k}_2+m}{s-m^2}+\frac{\not{k}_1-\not{k}_3+m}{u-m^2}\right)u(k_1,+)\ \left(\overline{v}(k_2,-)\mathcal{P}_R\xi\right),$$

$$2g^2\,\overline{u}(k_4,+)\left(\frac{m}{t-m^2}-\frac{\not{k}_1-\not{k}_3-m}{u-m^2}+\frac{\not{k}_1+\not{k}_2+m}{s-m^2}\right)\gamma^5 u(k_1,+)\ \left(\overline{v}(k_2,-)\mathcal{P}_R\xi\right).$$

There still remains to perform the SUSY transformation of the fermion in the initial state with the diagrams of the process $\phi^*(k_1)\overline{\Psi}(k_2,-)\to A(k_3)\Psi(k_4,+)$:

$$+ \qquad\qquad + \tag{5.36}$$

The relative (and, again, the global sign) of the second diagram results from the Wick theorem (here we only show the fermion contractions explicitly):

$$(-1)^2\cdot\langle 0|a\,b\,\left(\overline{\Psi}\Psi^c\phi^*\right)\left(\overline{\Psi^c}\Psi^c A\right)a^\dagger\,d^\dagger|0\rangle \tag{5.37}$$

The trick in this calculation is to disentangle the contractions by rewriting the second interaction operator,

$$\overline{\Psi}\Psi A\equiv\overline{\Psi}\Gamma\Psi A=\left(\overline{\Psi}\Gamma\Psi A\right)^T=(-1)\cdot\Psi^T\mathcal{C}^{-1}\left(\mathcal{C}\Gamma\mathcal{C}^{-1}\right)\Psi^c A\equiv(-1)^2\cdot\overline{\Psi^c}\Psi^c A \quad ,$$

because in this model only scalar, pseudoscalar or chiral scalar couplings appear that are invariant (i.e. their gamma matrices) under the charge conjugation transformation. One of the additional signs is due to the anticommutation of the Fermi field operators when transposing, the other stems from the relations

$$\mathcal{C}\overline{\Psi}^T=\Psi^c,\qquad \Psi^T\mathcal{C}^{-1}=-\overline{\Psi^c}\quad . \tag{5.38}$$

The sum of the last three diagrams results in:

$$\begin{aligned}-4g^2\cdot\overline{u}(k_4,+)\Bigg(&\frac{\not{k}_1+\not{k}_2+m}{s-m^2}\mathcal{P}_R-\mathcal{P}_R\frac{\not{k}_1-\not{k}_4-m}{t-m^2}\\&+\frac{2m}{u-m^2}\mathcal{P}_R\Bigg)u(k_2,-)\left(\overline{v}(k_1,+)\mathcal{P}_R\xi\right)\quad .\end{aligned}$$

Now we sum up the contributions of the several processes of this SWI separately for each of the reaction channels. A common prefactor $2g^2$ is suppressed in the following.

$$\begin{aligned}\textbf{s-channel}\propto\ &2m\big(\overline{v}(k_2,-)\mathcal{P}_L u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&-2\big(\overline{v}(k_2,-)\mathcal{P}_R u(k_1,+)\big)\big(\overline{u}(k_4,+)\not{k}_3\mathcal{P}_R\xi\big)\\&-3m\big(\overline{v}(k_2,-)u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&-m\big(\overline{v}(k_2,-)\gamma^5 u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&+\big(\overline{u}(k_4,+)(\not{k}_1+\not{k}_2+m)u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\&+\big(\overline{u}(k_4,+)(\not{k}_1+\not{k}_2+m)\gamma^5 u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\&-2\big(\overline{u}(k_4,+)(\not{k}_1+\not{k}_2+m)\mathcal{P}_R u(k_2,-)\big)\big(\overline{v}(k_1,+)\mathcal{P}_R\xi\big)\end{aligned} \tag{5.39}$$

$$
\begin{aligned}
\textbf{t-channel} \propto\ & 2\big(\overline{u}(k_4,+)\mathcal{P}_R u(k_1,+)\big)\big(\overline{v}(k_2,-)\not{k}_3\mathcal{P}_R\xi\big)\\
&- 2m\big(\overline{u}(k_4,+)\mathcal{P}_L u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\
&- \big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_4+m)u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\
&- \big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_4+m)\gamma^5 u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\
&+ 3m\big(\overline{u}(k_4,+)u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\
&+ m\big(\overline{u}(k_4,+)\gamma^5 u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\
&+ 2\big(\overline{u}(k_4,+)\mathcal{P}_R(\not{k}_1-\not{k}_4-m)u(k_2,-)\big)\big(\overline{v}(k_1,+)\mathcal{P}_R\xi\big)
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
\textbf{u-channel} \propto\ & -2\big(\overline{u}(k_4,+)\mathcal{P}_R u(k_2,-)\big)\big(\overline{v}(k_1,+)\not{k}_3\mathcal{P}_R\xi\big)\\
&- \big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_3+m)u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\
&+ \big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_3-m)\gamma^5 u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\
&+ \big(\overline{u}(k_4,+)(\not{k}_1-\not{k}_3+m)u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\
&- \big(\overline{u}(k_4,+)(\not{k}_1-\not{k}_3-m)\gamma^5 u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\
&- 4m\big(\overline{u}(k_4,+)\mathcal{P}_R u(k_2,-)\big)\big(\overline{v}(k_1,+)\mathcal{P}_R\xi\big)
\end{aligned}
\tag{5.41}
$$

The first, third and fourth line of (5.39) can be combined to give

$$-4m\big(\overline{v}_2\mathcal{P}_R u_1\big)\big(\overline{u}_4\mathcal{P}_R\xi\big)$$

(in the sequel we abbreviate $u(k_1,+)$ by u_1 etc.). Adding the second line from equation (5.39), we arrive at

$$-2\big(\overline{v}_2\mathcal{P}_R u_1\big)\big(\overline{u}_4\,(\not{k}_3+2m)\,\mathcal{P}_R\xi\big) \quad . \tag{5.42}$$

Adding the fifth and sixth line of (5.39) yields

$$2\big(\overline{v}_2\mathcal{P}_R\xi\big)\big(\overline{u}_4\,(\not{k}_1+\not{k}_2+m)\,\mathcal{P}_R u_1\big) \quad . \tag{5.43}$$

Applying the Fierz identities, we bring this expression and also the term of the last line in (5.39) into the form of (5.42). In the following calculation we use the notation $k_{12} \equiv k_1 + k_2$. The brackets indicate our fundamental spinors in spinor products of the Fierz identities. In contrast to the Fierz identities used for checking the Jacobi identities, there is no additional sign in here as there is only one anticommuting spinor.

$$
\begin{aligned}
2\big([\overline{u}_4(\not{k}_{12}+m)]\,\mathcal{P}_R u_1\big)\big(\overline{v}_2\,[\mathcal{P}_R\xi]\big) =\ & +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big)\\
&+\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma^5\mathcal{P}_R u_1\big)\\
&+\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\mathcal{P}_R u_1\big)\\
&+\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\gamma^5\mathcal{P}_R u_1\big)\\
&+\frac{1}{4}\big(\overline{u}_4(\not{k}_{12}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{v}_2\sigma_{\mu\nu}\mathcal{P}_R u_1\big)
\end{aligned}
\tag{5.44}
$$

By Fierzing, the last line of (5.39) can be written as

$$\begin{aligned}
-2\big([\overline{u}_4(\not{k}_{12}+m)]\,\mathcal{P}_R u_2\big)\big(\overline{v}_1\,[\mathcal{P}_R\xi]\big) = & -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_1\mathcal{P}_R u_2\big) \\
& -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{v}_1\gamma^5\mathcal{P}_R u_2\big) \\
& -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_1\gamma_\mu\mathcal{P}_R u_2\big) \\
& -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_1\gamma_\mu\gamma^5\mathcal{P}_R u_2\big) \\
& -\frac{1}{4}\big(\overline{u}_4(\not{k}_{12}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{v}_1\sigma_{\mu\nu}\mathcal{P}_R u_2\big) \quad .
\end{aligned} \tag{5.45}$$

To give the expressions a common structure we again use the rules of [?] to "turn round" the second term in parentheses on the right hand side of (5.45):

$$\begin{aligned}
(5.45) = & +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big) \\
& +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma^5\mathcal{P}_R u_1\big) \\
& -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\mathcal{P}_L u_1\big) \\
& +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\gamma^5\mathcal{P}_L u_1\big) \\
& -\frac{1}{4}\big(\overline{u}_4(\not{k}_{12}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{v}_2\sigma_{\mu\nu}\mathcal{P}_R u_1\big)
\end{aligned} \tag{5.46}$$

When adding (5.44) and (5.46) the tensor part vanishes. Absorbing the γ^5 matrices into the chiral projectors the vector contributions in (5.44) and (5.46) cancel the terms containing the axial vector, while the scalar and pseudoscalar contributions can be combined to give:

$$2\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big) \quad . \tag{5.47}$$

Summing up (5.42) and (5.47) yields the following result for the whole s-channel contributions

$$2\big(\overline{u}_4(\not{k}_1+\not{k}_2-\not{k}_3-m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big) = 2\big(\overline{u}_4(\not{k}_4-m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big) = 0 \quad . \tag{5.48}$$

In the analytical expression for the t-channel diagrams (5.40), combining the first two as well as the fifth and the sixth line gives

$$2\big(\overline{v}_2(\not{k}_3+2m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) \quad . \tag{5.49}$$

On the other hand, the third and fourth line yield

$$-2\big(\overline{v}_2(\not{k}_1-\not{k}_4+m)\mathcal{P}_R u_1\big)\big(\overline{u}_4\mathcal{P}_R\xi\big) \quad . \tag{5.50}$$

To perform the calculation in a more effective way, we manipulate the last line in (5.40), in particular we "turn round" the first term in parentheses,

$$+2\big(\overline{v}_2(\not{k}_1-\not{k}_4+m)\mathcal{P}_R v_4\big)\big(\overline{v}_1\mathcal{P}_R\xi\big) \quad . \tag{5.51}$$

It also has to be Fierz transformed, together with (5.50), to get the same spinor structure as (5.49). Again we use the notation $k_{14} \equiv k_1 - k_4$, the brackets distinguishing the spinors used as the fundamental ones in the Fierz identities. From (5.50) we obtain

$$\begin{aligned} -2\big([\overline{v}_2(\not{k}_{14}+m)]\,\mathcal{P}_R u_1\big)\big(\overline{u}_4\,[\mathcal{P}_R\xi]\big) = &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) \\ &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_1\big) \\ &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\mathcal{P}_R u_1\big) \\ &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\gamma^5\mathcal{P}_R u_1\big) \\ &-\frac{1}{4}\big(\overline{v}_2(\not{k}_{14}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_1\big) \end{aligned} \tag{5.52}$$

For the Fierz transformation of (5.51) we "turn round" the product containing the spinors $\overline{v}_1$ and v_4, getting the spinors $\overline{u}_4$ and u_1.

$$\begin{aligned} 2\big([\overline{v}_2(\not{k}_{14}+m)]\,\mathcal{P}_R v_4\big)\big(\overline{v}_1\,[\mathcal{P}_R\xi]\big) = &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) \\ &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_1\big) \\ &+\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\mathcal{P}_L u_1\big) \\ &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\gamma^5\mathcal{P}_L u_1\big) \\ &+\frac{1}{4}\big(\overline{v}_2(\not{k}_{14}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_1\big) \end{aligned} \tag{5.53}$$

As was the case for the s-channel, the tensor contributions to (5.52) and (5.53) cancel out, while in each equation the vector part again cancels the axial vector. The scalar and pseudoscalar parts from both Fierz transformations give

$$+2\big(\overline{v}_2(\not{k}_4-\not{k}_1-m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) \quad , \tag{5.54}$$

so finally the result for the t-channel is written as:

$$2\big(\overline{v}_2(\not{k}_3+\not{k}_4-\not{k}_1+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) = 2\big(\overline{v}_2(\not{k}_2+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) = 0 \tag{5.55}$$

The same calculation goes through for the u-channel, transferring (5.41):

$$\begin{aligned} (5.41) = &-2\big(\overline{v}_1(\not{k}_3+2m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big) \\ &+2\big(\overline{v}_1(\not{k}_2-\not{k}_4+m)\mathcal{P}_R u_2\big)\big(\overline{u}_4\mathcal{P}_R\xi\big) \\ &-2\big(\overline{v}_1(\not{k}_2-\not{k}_4+m)\mathcal{P}_R v_4\big)\big(\overline{v}_2\mathcal{P}_R\xi\big) \end{aligned} \tag{5.56}$$

The Fierz transformations of the last two lines (again we "invert" the products containing $\overline{v}_2$ and v_4 in the third line and abbreviate $k_2 - k_4$ by k_{24}) are:

$$\begin{aligned} 2\big([\overline{v}_1(\not k_{24}+m)]\,\mathcal{P}_R u_2\big)\big(\overline{u}_4\,[\mathcal{P}_R\xi]\big) = \; & \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big) \\ & + \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_2\big) \\ & + \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\mathcal{P}_R u_2\big) \\ & + \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\gamma^5\mathcal{P}_R u_2\big) \\ & + \frac{1}{4}\big(\overline{v}_1(\not k_{24}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_2\big) \end{aligned} \tag{5.57}$$

$$\begin{aligned} -2\big([\overline{v}_1(\not k_{24}+m)]\,\mathcal{P}_R v_4\big)\big(\overline{v}_2\,[\mathcal{P}_R\xi]\big) = \; & \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big) \\ & + \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_2\big) \\ & - \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\mathcal{P}_R v_4\big) \\ & - \frac{1}{2}\big(\overline{v}_1(\not k_{24}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\gamma^5\mathcal{P}_R v_4\big) \\ & - \frac{1}{4}\big(\overline{v}_1(\not k_{24}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_2\big) \end{aligned} \tag{5.58}$$

The vector contributions as well as the axial vector parts vanish separately for each process as in the s- and t-channels, while the tensor contributions of (5.57) and (5.58) cancel each other. The scalar and pseudoscalar contributions are equal und sum up to

$$2\big(\overline{v}_1(\not k_{24}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big). \tag{5.59}$$

Therefore the result of (5.56) is

$$2\big(\overline{v}_1(\not k_2-\not k_3-\not k_4-m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big) = -2\big(\overline{v}_1(\not k_1+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big) = 0 \quad . \tag{5.60}$$

So finally we can see that s-, t- and u-channel diagrams vanish separately and we find the SWIs of $(2 \to 2)$ processes containing two as well as four fermions to be fulfilled.

Chapter 6

The O'Raifeartaigh model

6.1 Spontaneous breaking of Supersymmetry

The simplest model in which supersymmetry is spontaneously broken is the O'Raifeartaigh model. To be more precise it is a whole class of models (cf. [?]), the particular O'Raifeartaigh model being only a special case. The particle content, some special remarks and the Feynman rules of the O'Raifeartaigh model (from hereon referred to as the OR model) are collected in the appendix. As was proven by O'Raifeartaigh, at least three chiral superfields are needed to make spontaneous supersymmetry breaking possible.

This model offers the opportunity to examine what happens to the SWI in the case of spontaneous breaking. Of course, the derivation of identity (3.2) breaks down together with our symmetry since the vacuum is no longer left invariant by the action of the supercharge. But we want to show an example of an SWI, in the sense, that we calculate a SWI as if (3.2) were still valid and take a look at the terms violating the SWI. The latter should turn out to be proportional to the parameters of SUSY breaking.

6.2 Preliminaries to the O'Raifeartaigh model

For the OR model as a spontaneously broken supersymmetric model the relation

$$Q\,|0\rangle = 0 \tag{6.1}$$

is no longer fulfilled, but this had to be postulated to be able to derive the SWI. This section will show what happens to the SWI if we were to assume (6.1) to be valid anyhow.

There is a higher number of particles in the OR model than in previously considered models. We gratefully make use of this fact as the number of participating diagrams in an SWI shrinks enormously with a growing variety of external particles. Unfortunately this advantage is partly lost since up to three different scalar particles appear as a result of the SUSY transformations of fermionic annihilation and creation operators.

With the experience from last chapter's toy model we can immediately write down the transformation laws of the annihilators (and therefore also for the creators).

First of all we want to introduce a common notation for all particles: The annihilators of the scalars are denoted by a_A, a_B, $a_\pm^\phi$ and $a_\pm^\Phi$, the Majorana fermion's annihilator by c, while the annihilators for the Dirac fermion are denoted by b and d as usual. The creators are the Hermitean adjoints, of course.

As for the toy model, the fermionic partner of the scalar field which is split into real and imaginary parts, gives the lefthanded component of a Dirac fermion so we can directly take over the result (5.11):

$$\boxed{\begin{aligned}[][Q(\xi), a_A(k)] &= \mathrm{i}\sum_\sigma \Big(\left(\overline{\xi}\mathcal{P}_L u(k,\sigma)\right) b(k,\sigma) + \left(\overline{\xi}\mathcal{P}_R u(k,\sigma)\right) d(k,\sigma) \Big) \\ [Q(\xi), a_B(k)] &= \sum_\sigma \Big(\left(\overline{\xi}\mathcal{P}_L u(k,\sigma)\right) b(k,\sigma) - \left(\overline{\xi}\mathcal{P}_R u(k,\sigma)\right) d(k,\sigma) \Big)\end{aligned}} \tag{6.2}$$

The fermionic partner for the complex scalar field from the third superfield and its Hermitean adjoint are the righthanded component of that Dirac spinor. Consequently we can maintain (5.5) and (5.8),

$$\boxed{\left[Q(\xi), a_+^\Phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\overline{\xi}\mathcal{P}_R u(k,\sigma)\right) b(k,\sigma)} \quad , \tag{6.3}$$

$$\boxed{\left[Q(\xi), a_-^\Phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\overline{\xi}\mathcal{P}_L u(k,\sigma)\right) d(k,\sigma)} \quad . \tag{6.4}$$

In the case of the scalar field ϕ – the scalar component of the first superfield and superpartner of the Goldstino – we just have to set the two annihilators b and d equal to the Majorana annihilator c:

$$\boxed{\left[Q(\xi), a_+^\phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\overline{\xi}\mathcal{P}_R u(k,\sigma)\right) c(k,\sigma)} \quad , \tag{6.5}$$

$$\boxed{\left[Q(\xi), a_-^\phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\overline{\xi}\mathcal{P}_L u(k,\sigma)\right) c(k,\sigma)} \quad . \tag{6.6}$$

The transformations of the Dirac annihilators are analogous to (5.15) and (5.16), respectively:

$$\boxed{[Q(\xi), b(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k)\mathcal{P}_R + \mathrm{i}a_B(k)\mathcal{P}_R + \sqrt{2}a_+^\Phi(k)\mathcal{P}_L\Big)\xi} \quad , \tag{6.7}$$

$$\boxed{[Q(\xi), d(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k)\mathcal{P}_L - \mathrm{i}a_B(k)\mathcal{P}_L + \sqrt{2}a_-^\Phi(k)\mathcal{P}_R\Big)\xi} \quad . \tag{6.8}$$

For the first superfield we use equation (3.10) und get

$$\boxed{[Q(\xi), c(k,\sigma)] = -\mathrm{i}\sqrt{2}\,\overline{u}(k,\sigma)\Big(a_-^\phi\mathcal{P}_R + a_+^\phi\mathcal{P}_L\Big)\xi} \tag{6.9}$$

6.3 Example for an SWI in the OR model

As for the WZ model before, we want to construct an example for an SWI. Again we start with a string of fields in which the spin of initial and final states differ by half a unit. As mentioned above, we want to make use of the greater variety of particles available in this model.

Our choice for an example is the following:

$$\begin{aligned}0 \neq & \left\langle 0 \middle| \left[Q(\xi), a_A(k_3) c(k_4,+) a_-^{\Phi\,\dagger}(k_1) a_+^{\phi\,\dagger}(k_2) \right] \middle| 0 \right\rangle \\ = \quad & \mathrm{i} \sum_\sigma \left(\overline{\xi} \mathcal{P}_L u(k_3,\sigma)\right) \cdot \mathcal{M}\big(\Phi(k_1)\phi^*(k_2) \to \Psi(k_3,\sigma)\chi(k_4,+)\big) \\ & + \mathrm{i} \sum_\sigma \left(\overline{\xi} \mathcal{P}_R u(k_3,\sigma)\right) \cdot \mathcal{M}\big(\Phi(k_1)\phi^*(k_2) \to \overline{\Psi}(k_3,\sigma)\chi(k_4,+)\big) \\ & + \mathrm{i}\sqrt{2} \sum_\sigma \left(\overline{u}(k_1,\sigma) \mathcal{P}_L \xi\right) \cdot \mathcal{M}\big(\overline{\Psi}(k_1,\sigma)\phi^*(k_2) \to A(k_3)\chi(k_4,+)\big)\end{aligned} \tag{6.10}$$

The processes resulting from the SUSY transformations of the Majorana fermion in the final state and the massless boson in the initial state do not contribute. For the transformation of the remaining particles we write down only the nonvanishing terms. The first process with two diagrams

$$+ \qquad , \tag{6.11}$$

produces, after multiplication with the appropriate prefactor, the analytical expression

$$2g^2 m \cdot \left(\overline{\xi} \mathcal{P}_L \not{k}_3 v(k_4,+)\right) \cdot \left(\frac{1}{s - m^2 + 2\lambda g} - \frac{1}{s - m^2 - 2\lambda g} \right) \quad . \tag{6.12}$$

The second process is analogous:

$$+ \tag{6.13}$$

The result is

$$-2g^2 m \cdot \left(\overline{\xi} \mathcal{P}_R \not{k}_3 v(k_4,+)\right) \cdot \left(\frac{1}{s - m^2 + 2\lambda g} + \frac{1}{s - m^2 - 2\lambda g} \right) . \tag{6.14}$$

There exists just one diagram for the third process,

$$. \tag{6.15}$$

Here we have to keep an eye on the signs again, while having to apply the Wick theorem. The resulting amplitude is

$$+4g^2 m \cdot \left(\overline{\xi} \mathcal{P}_R (\not{k}_1 + \not{k}_2) v(k_4,+)\right) \cdot \frac{1}{s - m^2} = +4g^2 m \cdot \left(\overline{\xi} \mathcal{P}_R \not{k}_3 v(k_4,+)\right) \cdot \frac{1}{s - m^2} \tag{6.16}$$

Implicitly we used momentum conservation and the Dirac equation for $v(k_4)$, which simply is $\not{k}_4 v(k_4) = 0$ for the Majorana fermion being the Goldstino.

When choosing special transformation spinors ξ, we see that the righthanded and lefthanded part of the identity must be fulfilled separately. As is immediately seen the SWI is violated as we expected from the beginning. Inspecting the limit $\lambda \to 0$ shows that the contribution containing the lefthanded chiral projector vanishes and the parts with the righthanded chiral projectors cancel each other. This is understandable by remembering that the parameter λ controls the spontaneous symmetry breaking of the OR model as it produces the mass splitting between the particles of the second and the third superfield, which came from diagonalizing the mass terms.

This violation of the SWI of the type derived in [?] and [?] stems from the non-invariance of the vacuum under SUSY transformations in spontaneously broken SUSY theories. It can be avoided by using a formalism based on the concept of a conserved Noether current for the supersymmetry; this will be shown in the next part.

Part II

SUSY Ward identities via the current

Chapter 7

The supersymmetric current and SWI

There are some inherent problems in the method of calculating SWIs the way presented in the last part: It does not work for spontaneously broken supersymmetry and is also only applicable for on-shell identities. To develop stringent tests for supersymmetric field theories, it will prove useful to consider off-shell identities as well, as much more of the underlying physics is involved in such relations. In this part we will first present how SWI can be implemented when using the current of the supersymmetry and then show examples for the Wess-Zumino model. To verify that this method is also valid for spontaneously broken supersymmetry, we extend our calculations to the O'Raifeartaigh model. Afterwards we turn to the combination of (global) supersymmetry and gauge symmetries when examining currents in supersymmetric Yang-Mills theories. This is important because realistic models should, of course, incorporate at least the gauge symmetries of the Standard Model.

7.1 Ward identities – current vs. external states

In this section we describe the connection between the SWI in the formalism derived in [?] and [?] and similar relations which can be obtained with the help of supersymmetric current conservation. The name "supersymmetric" current is a bit misleading as this current is not invariant under SUSY transformations. In fact, the current mentioned here is closely related to a spinor component of a real superfield provided with an additional vector index, called the supercurrent (cf. [?], [?]). The scalar component of the supercurrent is the current of R symmetry, while the vector component is given by the energy-momentum tensor. The supersymmetric current has the Lorentz transformation properties of a vectorspinor. In a local version of supersymmetry – supergravity – the corresponding gauge field is the gravitino.

To derive this kind of SWI we write down a time-ordered product of a string of field operators (appearing in the supersymmetric model under consideration) with the operator insertion of the supersymmetric current,

$$\langle 0|\mathrm{T}\left[\mathcal{J}^{\mu}(x)\mathcal{O}_1(y_1)\mathcal{O}_2(y_2)\ldots\mathcal{O}_n(y_n)\right]|0\rangle \tag{7.1}$$

Taking the derivative of this expression with respect to x^μ (we use the abbreviation $\partial^x_\mu \equiv \partial/\partial x^\mu$), we get:

$$\begin{aligned}
&\mathrm{i}\partial^x_\mu \left\langle 0\middle|\mathrm{T}\left[\mathcal{J}^\mu(x)\mathcal{O}_1(y_1)\ldots\mathcal{O}_n(y_n)\right]\middle|0\right\rangle \\
&\quad= \sum_{i=1}^{n}\chi_i \left\langle 0\middle|\mathrm{T}\left[\mathcal{O}_1\ldots\mathcal{O}_{i-1}\left[\mathrm{i}\mathcal{J}^0(x),\mathcal{O}_i(y_i)\right]_{P_i}\delta(x^0-y_i^0)\,\mathcal{O}_{i+1}\ldots\mathcal{O}_n\right]\middle|0\right\rangle \\
&\qquad\qquad + \mathrm{i}\left\langle 0\middle|\mathrm{T}\left[\partial^x_\mu\mathcal{J}^\mu(x)\mathcal{O}_1(y_1)\ldots\mathcal{O}_n(y_n)\right]\middle|0\right\rangle
\end{aligned}\tag{7.2}$$

Here χ_i has the meaning of a sign prefactor

$$\chi_i \equiv (-1)^{\sum_{j=1}^{i-1}P_j}, \tag{7.3}$$

which arises by anticommuting the Grassmann odd current with Fermi field operators. P is the Grassmann parity of the fields, 1 for fermions and 0 for bosons. In the same manner we have introduced the graded commutator

$$[A,B]_{P=1} \equiv \{A,B\} \text{ for fermions,} \quad [A,B]_{P=0} \equiv [A,B] \text{ otherwise} \tag{7.4}$$

as an anticommutator in the case of two fermionic operators and a commutator in all other cases.

The last term in (7.2), which is created by applying the derivative to the current, vanishes due to current conservation. The terms with the graded commutators arise when acting with the time derivative on the step functions in the time ordered product. We make use of the fact that the equal time commutator (or anticommutator in the case of a fermionic operator) of the zero component of the current with an operator (for instance, the field operator of the fundamental fields of the theory) equals the symmetry transformation (in our case the SUSY transformation) of the considered field:

$$\left[\mathrm{i}\bar{\xi}\mathcal{J}^0(x),\mathcal{O}(y)\right]\delta(x^0-y^0) = \delta_\xi\mathcal{O}(y)\cdot\delta^4(x-y) \tag{7.5}$$

With the help of this relation we can rewrite the right hand side of (7.2). Furthermore we switch to momentum space and replace the spacetime derivative acting on the left hand side of equation (7.2) by the momentum k_μ which flows into the Green function through the current operator insertion (so $-k^\mu = \sum_i p_i^\mu$ is the sum over the incoming momenta of all other external legs).

$$\begin{aligned}
&k_\mu \mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\bar{\xi}\mathcal{J}^\mu(x)\mathcal{O}_1(y_1)\ldots\mathcal{O}_n(y_n)\right]\middle|0\right\rangle \\
&\qquad= \sum_{i=1}^{n}\mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\mathcal{O}_1\ldots\mathcal{O}_{i-1}\left(\delta_\xi\mathcal{O}_i(y_i)\right)\mathcal{O}_{i+1}\ldots\mathcal{O}_n\right]\middle|0\right\rangle\cdot\delta^4(x-y_i)
\end{aligned}\tag{7.6}$$

In (7.6) the supersymmetric current has been multiplied by the SUSY transformation parameter ξ and hence became a bosonic operator. There are two consequences: we could forget about the sign prefactor which was part of (7.2) and all graded commutators became commutators. In (7.5) and (7.6) we used the usual notation for the SUSY transformations of the fields (with transformation parameter ξ).

At tree level the identity (7.2) is valid for linearly as well as nonlinearly realized symmetries both for on-shell and off-shell processes (cf. for instance the path integral derivation of the Ward identities in [?]). In the case of nonlinearly realized symmetries, not only higher than quadratic terms will appear in the current operator but also composite operators in the transformations of the fields. To put the identity (7.2) on the mass shell we have to apply the LSZ reduction formula [?], [?], [?] to all external legs except the current itself, which remains unamputated:

We used the abbreviation $f_i \equiv e^{-\mathrm{i}k_i x_i}/\sqrt{(2\pi)^3 2k_i^0}$. For simplicity we denoted only the amputation procedure for bosons. The big grey blob stands for the process under consideration (i.e. the interaction operators needed to connect the external fields in (7.2)), while the smaller blob will become our standard convention for a current insertion. On shell, all the so called contact terms on the right hand side of equation (7.2) vanish. This is seen by inspection of the amputation procedure for those Green functions with the transformed fields: Let the external particle corresponding to the ith field have momentum p_i on the left hand side, then on the right hand side the particle corresponding to the transformed field has its momentum increased by the momentum influx through the current $p_i + k$. For the sake of simplicity, we show an example involving only scalar fields:

$$D_F(p_i)^{-1} \cdot D_F(p_i + k) = \frac{p_i^2 - m_i^2}{(p_i + k)^2 - m_i^2} \tag{7.7}$$

These two propagator factors do not cancel like all other propagators of external particles do, hence when setting the external momenta $p_j, j = 1, 2, \ldots$ on the mass shell, this yields zero for every term on the right hand side.

Another interesting phenomenon happens for spontaneously broken symmetries, where a field gets a vacuum expectation value and is therefore shifted by a constant. A term linear in the field appears in the current, or more precisely, a term proportional to the derivative of the Goldstone boson. This contributes tadpole-like diagrams which, if resummed, shift the appropriate poles of the fields according to the mass splitting from the spontaneous symmetry breaking. Since coupling constants and vacuum expectation values are combined to yield masses of particles, there is a mixing of different orders in perturbation theory contributing to the Ward identity. For supersymmetric field theories the corresponding term in the current is given by a gamma matrix times the derivative of the Goldstino field. We will study this in detail in the O'Raifeartaigh model below.

7.2 Simplest example – Wess-Zumino model

Like any continous symmetry in a field theory, supersymmetry possesses a conserved current whose charge is the generator of the symmetry transformation. Supersymmetry is no symmetry of the Lagrangean density but only of the action. It transforms the Lagrangean density into a total derivative which vanishes upon integration over space-time. The following discussion is similar to that in [?]. If we assume that the change of the Lagrangean density under a SUSY transformation takes on the form

$$\delta_\xi \mathcal{L} = \overline{\xi} \partial_\mu K^\mu, \tag{7.8}$$

we can calculate the structure of K^μ. We want to derive the supersymmetric current for the WZ model as this is the simplest supersymmetric model. In the Lagrangean density only the D-term of the kinetic part and the F-terms from the superpotential appear. The SUSY transformation of a D-term of an arbitrary superfield is given by [?], [?] [1]

$$\delta_\xi D = \overline{\xi}\gamma^5 \partial\!\!\!/ \lambda \quad . \tag{7.9}$$

Here λ is a spinor being the θ^3 coefficient in a superspace expansion of a general superfield. We conclude, that for the kinetic part of the WZ model Lagrangean density as a product of a right- and a lefthanded chiral superfield, $\hat{\Phi}^\dagger\hat{\Phi}$,

$$\delta_\xi \mathcal{L}_{\mathrm{kin}} = \delta_\xi \left[\frac{1}{2}\hat{\Phi}^\dagger\hat{\Phi}\right]_D = \overline{\xi}\gamma^5 \partial\!\!\!/ \frac{1}{2}\left[\hat{\Phi}^\dagger\hat{\Phi}\right]_\lambda \quad . \tag{7.10}$$

The appropriate λ can be read off from equation (26.2.24) in [?] or, in our conventions, from equation (5.116) in [?], by taking into consideration that the general superfield $\hat{\Phi}_1$ there is to be set to the right chiral superfield $\hat{\Phi}^\dagger$ and the second superfield $\hat{\Phi}_2$ to the Hermitean adjoint left chiral superfield $\hat{\Phi}$. This enables us to make the following replacements (of course, the SUSY transformation can be done by brute force in a component language but the superfield formalism is much more elegant) [2]:

$$\begin{array}{ll} \lambda_1 \equiv 0 & \lambda_2 \equiv 0 \\ V_1^\mu \equiv -\mathrm{i}\partial^\mu\phi^* & V_2^\mu \equiv \mathrm{i}\partial^\mu\phi \\ C_1 \equiv \phi^* & \omega_1 \equiv \mathrm{i}\sqrt{2}\Psi_R \\ C_2 \equiv \phi & \omega_2 \equiv -\mathrm{i}\sqrt{2}\Psi_L \\ N_1 \equiv F^* & M_1 \equiv -\mathrm{i}F^* \\ N_2 \equiv F & M_2 \equiv \mathrm{i}F \end{array} \tag{7.11}$$

The result is

$$K^\mu_{\mathrm{kin}} = \frac{1}{\sqrt{2}}\gamma^\mu\Big((\partial\!\!\!/\phi)\,\Psi_R + (\partial\!\!\!/\phi^*)\,\Psi_L - \mathrm{i}F\Psi_R - \mathrm{i}F^*\Psi_L \Big) \quad . \tag{7.12}$$

From the transformation of the superpotential's F-terms we write down the relation

$$\delta_\xi \mathcal{L}_{\mathrm{pot}} = \delta_\xi \left[\frac{m}{2}\hat{\Phi}^2 + \frac{\lambda}{3!}\hat{\Phi}^3\right]_F + \mathrm{h.c.} = -\mathrm{i}\sqrt{2}\,\overline{\xi}\mathcal{P}_L \partial\!\!\!/ \left[\frac{m}{2}\hat{\Phi}^2 + \frac{\lambda}{3!}\hat{\Phi}^3\right]_\psi + \mathrm{h.c.} \tag{7.13}$$

The contribution from the potential becomes

$$K^\mu_{\mathrm{pot}} = -\mathrm{i}\sqrt{2}\gamma^\mu\Big(m\Psi_L\phi + m\Psi_R\phi^* + \frac{1}{2}\lambda\Psi_L\phi^2 + \frac{1}{2}\lambda\Psi_R(\phi^*)^2 \Big) \tag{7.14}$$

(NB: Herein λ is the coupling constant of the WZ model, not a spinor component of a superfield.) So altogether we get for this contribution to the supersymmetric current

[1] The relative factor of i between both references comes from the different conventions concerning the metric and hence the gamma matrices.

[2] In the appendix a detailed derivation for the supersymmetric current in supersymmetric Yang-Mills theories can be found.

$$
\begin{aligned}
K^\mu = \frac{1}{\sqrt{2}}\gamma^\mu \Big(& (\partial\!\!\!/\phi)\,\Psi_R + (\partial\!\!\!/\phi^*)\,\Psi_L - \mathrm{i}F\Psi_R - \mathrm{i}F^*\Psi_L - 2m\mathrm{i}\Psi_L\phi \\
& - 2m\mathrm{i}\Psi_R\phi^* - \mathrm{i}\lambda\Psi_L\phi^2 - \mathrm{i}\lambda\Psi_R(\phi^*)^2 \Big)
\end{aligned}
\tag{7.15}
$$

Inserting the definitions of the fields A, B, $\mathcal{F}$ and $\mathcal{G}$ yields

$$
\begin{aligned}
K^\mu = & \frac{1}{2}\gamma^\mu(\partial\!\!\!/A)\Psi - \frac{\mathrm{i}}{2}\gamma^\mu\gamma^5(\partial\!\!\!/B)\Psi - \frac{\mathrm{i}}{2}\gamma^\mu\mathcal{F}\Psi - \frac{1}{2}\gamma^\mu\gamma^5\mathcal{G}\Psi \\
& - \mathrm{i}m\gamma^\mu A\Psi - m\gamma^\mu\gamma^5 B\Psi - \frac{\mathrm{i}\lambda}{2\sqrt{2}}\gamma^\mu\left(A^2 - B^2\right)\Psi - \frac{\lambda}{\sqrt{2}}\gamma^\mu\gamma^5 AB\Psi
\end{aligned}
\tag{7.16}
$$

The so called Noether part of the supersymmetric current (by which the current is given in the case of an invariant Lagrangean density) reads

$$
\sum_{\text{all fields}} \frac{\partial_R\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_\xi\Phi = -\overline{\xi}N^\mu \quad . \tag{7.17}
$$

In the WZ models these derivatives are

$$
\frac{\partial_R\mathcal{L}}{\partial(\partial_\mu A)} = \partial^\mu A, \qquad \frac{\partial_R\mathcal{L}}{\partial(\partial_\mu B)} = \partial^\mu B, \qquad \frac{\partial_R\mathcal{L}}{\partial(\partial_\mu \Psi)} = \frac{\mathrm{i}}{2}\overline{\Psi}\gamma^\mu, \tag{7.18}
$$

while the SUSY transformations of the several fields are stated in (2.5). The Noether part therefore is

$$
N^\mu = -\left(\partial^\mu A\right)\Psi - \mathrm{i}\left(\partial^\mu B\right)\gamma^5\Psi - \frac{1}{2}\left[\partial\!\!\!/\left(A - \mathrm{i}\gamma^5 B\right)\right]\gamma^\mu\Psi + \frac{\mathrm{i}}{2}\left(\mathcal{F} + \mathrm{i}\gamma^5\mathcal{G}\right)\gamma^\mu\Psi \tag{7.19}
$$

Adding the two parts (7.16) and (7.17) results in the supersymmetric current for the WZ model

$$
\boxed{
\begin{aligned}
\mathcal{J}^\mu &= K^\mu + N^\mu \\
&= \mathrm{i}\left((\mathrm{i}\partial\!\!\!/ - m)A\right)\gamma^\mu\Psi + \left((\mathrm{i}\partial\!\!\!/ + m)B\right)\gamma^5\gamma^\mu\Psi \\
&\quad - \frac{\mathrm{i}\lambda}{2\sqrt{2}}\gamma^\mu\left(A^2 - B^2\right)\Psi - \frac{\lambda}{\sqrt{2}}\gamma^\mu\gamma^5 AB\Psi
\end{aligned}
}
\tag{7.20}
$$

Now we can check – even if it is a little bit cumbersome – the current conservation explicitly.

$$
\begin{aligned}
\partial_\mu\mathcal{J}^\mu = & -(A)\Psi - \mathrm{i}mA(\partial\!\!\!/\Psi) - \underline{(\partial\!\!\!/A)(\partial\!\!\!/\Psi)} - \mathrm{i}\underline{m(\partial\!\!\!/A)\Psi} - \mathrm{i}(B)\gamma^5\Psi + mB\gamma^5(\partial\!\!\!/\Psi) \\
& + \underline{\mathrm{i}(\partial\!\!\!/B)\gamma^5(\partial\!\!\!/\Psi)} - \underline{m(\partial\!\!\!/B)\gamma^5\Psi} - \frac{\mathrm{i}\lambda}{2\sqrt{2}}\left(A^2 - B^2\right)\partial\!\!\!/\Psi - \underline{\frac{\mathrm{i}\lambda}{\sqrt{2}}(\partial\!\!\!/A)A\Psi} \\
& + \underline{\frac{\mathrm{i}\lambda}{\sqrt{2}}(\partial\!\!\!/B)B\Psi} + \frac{\lambda}{\sqrt{2}}\gamma^5 AB\partial\!\!\!/\Psi + \underline{\frac{\lambda}{\sqrt{2}}\gamma^5(\partial\!\!\!/A)B\Psi} + \underline{\frac{\lambda}{\sqrt{2}}\gamma^5 A(\partial\!\!\!/B)\Psi} \\
= & \frac{\lambda}{2\sqrt{2}}(\overline{\Psi}\Psi)\Psi - m\mathcal{F}\Psi - \frac{\lambda}{\sqrt{2}}A\mathcal{F}\Psi - \frac{\lambda}{\sqrt{2}}B\mathcal{G}\Psi + \frac{\lambda}{2\sqrt{2}}(\overline{\Psi}\gamma^5\Psi)\gamma^5\Psi - \mathrm{i}m\mathcal{G}\gamma^5\Psi \\
& + \frac{\mathrm{i}\lambda}{\sqrt{2}}B\mathcal{F}\gamma^5\Psi - \frac{\mathrm{i}\lambda}{\sqrt{2}}A\mathcal{G}\gamma^5\Psi + \mathrm{i}\mathcal{F}(\partial\!\!\!/\Psi) - \mathcal{G}\gamma^5(\partial\!\!\!/\Psi)
\end{aligned}
$$

The underlined terms cancel due to the equation of motion of the Majorana field Ψ. In the second equality the first eight terms stem from the equations of motion for the scalar fields A and B, while the last two come from inserting the equations of motion for the spinor field into the terms not underlined. The terms linear in $\mathcal{F}$ and $\mathcal{G}$ can be combined to give the equations of motion for the Majorana field and we are left with the trilinear fermion terms. Noting that third powers of Grassmann odd two component spinors $(\psi\psi)\psi$ vanish, the calculation

$$\begin{aligned} \left(\overline{\Psi}\Psi\right)\Psi &= (\psi\psi + \bar{\psi}\bar{\psi}) \cdot \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} (\bar{\psi}\bar{\psi})\psi \\ (\psi\psi)\bar{\psi} \end{pmatrix} \\ \left(\overline{\Psi}\gamma^5\Psi\right)\gamma^5\Psi &= (-\psi\psi + \bar{\psi}\bar{\psi}) \cdot \begin{pmatrix} -\psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} -(\bar{\psi}\bar{\psi})\psi \\ -(\psi\psi)\bar{\psi} \end{pmatrix} \end{aligned}, \tag{7.21}$$

shows the cancellation of the trilinear fermion terms. This finishes the proof of the desired current conservation:

$$\boxed{\partial_\mu \mathcal{J}^\mu = 0} \tag{7.22}$$

The current for a general model with an arbitrary number of superfields and the proof for its conservation can be found in appendix **??**.

Chapter 8

SWI via the current

8.1 Starting point: WZ model

In this section we want to calculate supersymmetric Ward identities (SWI) for the WZ model obtained with the help of the current as constructed in the previous chapter. The current for the WZ model is given by (7.20). We will show an example for an on-shell identity with three external particles (SWI with two external particles are just given by the propagators of the theory in the contact terms and are rather trivial) as well as for an off-shell SWI with the same number of external particles.

For the on-shell example, where the contact terms are absent, we choose a $(2 \to 1)$ process with two incoming scalar particles A, one outgoing fermion Ψ and a current insertion, to which (in lowest order perturbation theory) four different diagrams contribute:

$$+ \qquad\qquad + \qquad\qquad +$$

The momenta of the incoming As are denoted by k_1 and k_2 while the outgoing Majorana fermion's momentum is k'. The analytical expressions for the four diagrams are (from right to left):

$$(1) \qquad -\frac{\mathrm{i}\lambda}{\sqrt{2}}\gamma^\mu v(k'), \tag{8.1a}$$

$$(2) \qquad +\frac{3\mathrm{i}m\lambda}{\sqrt{2}\,(s-m^2)}\,(\not{k}_1+\not{k}_2-m)\,\gamma^\mu v(k'), \tag{8.1b}$$

$$(3) \qquad +\frac{\mathrm{i}\lambda}{\sqrt{2}\,(t-m^2)}\,(\not{k}_1-m)\,\gamma^\mu\,(\not{k}_2-\not{k}'+m)\,v(k'), \tag{8.1c}$$

$$(4) \qquad +\frac{\mathrm{i}\lambda}{\sqrt{2}\,(u-m^2)}\,(\not{k}_2-m)\,\gamma^\mu\,(\not{k}_1-\not{k}'+m)\,v(k'). \tag{8.1d}$$

For this problem the Mandelstam variables are

$$s \equiv (k_1+k_2)^2, \qquad t \equiv (k_2-k')^2, \qquad u \equiv (k_1-k')^2.$$

The verification of the SWI only needs the use of the Dirac equation $(\not{k}' + m)\, v(k') = 0$ and the relation $\not{k}\not{k} = k^2$. Applying the 4-gradient to the above matrix element produces the following sum that can be easily confirmed to be zero:

$$\begin{aligned}
\partial_\mu \langle\Psi|\mathcal{J}^\mu|AA\rangle = \frac{\lambda}{\sqrt{2}}\Bigg[&(\not{k}_1 + \not{k}_2 + m) - \frac{3m}{s-m^2}(\not{k}_1 + \not{k}_2 - m)(\not{k}' - \not{k}_1 - \not{k}_2) \\
&- \frac{(\not{k}_1 - m)(\not{k}' - \not{k}_2 - \not{k}_1)(\not{k}' - \not{k}_2 - m)}{t-m^2} \\
&- \frac{(\not{k}_2 - m)(\not{k}' - \not{k}_1 - \not{k}_2)(\not{k}' - \not{k}_1 - m)}{u-m^2}\Bigg] v(k') \\
= \frac{\lambda}{\sqrt{2}}\Bigg[&(\not{k}_1 + \not{k}_2 + m) - 3m - \frac{(\not{k}_1 - m)(t + m\not{k}_1)}{t-m^2} \\
&- \frac{(\not{k}_1 - m)(\not{k}_1 + m)(\not{k}_2 - \not{k}')}{t-m^2} - \frac{(\not{k}_2 - m)(u + m\not{k}_2)}{u-m^2} \\
&- \frac{(\not{k}_2 - m)(\not{k}_2 + m)(\not{k}_1 - \not{k}')}{u-m^2}\Bigg] v(k') \\
= \frac{\lambda}{\sqrt{2}}\Bigg[&m + \not{k}_1 + \not{k}_2 - 3m - \not{k}_2 + m - \not{k}_1 + m\Bigg] v(k') \quad = \quad 0 \quad \checkmark
\end{aligned} \tag{8.2}$$

Concerning (nonlinear) transformations, on-shell only the one-particle pole contributes. But for off-shell Ward identities the nonlinear terms give nonvanishing contributions in contact terms. The correct method to handle that difficulty is to define local operator insertions for every nonlinear term appearing in the transformations.

As an example for an off-shell identity we take the insertion of an A, a B and a Ψ field as the left hand side in (7.6)

$$\begin{aligned}
&\text{F.T.}\,\langle 0|\mathrm{T}\overline{\mathcal{J}_\mu}(y)\xi A(x_1)B(x_2)\Psi(x_3)|0\rangle = \\
&\quad \frac{\mathrm{i}}{p_1^2 - m^2}\frac{\mathrm{i}}{p_2^2 - m^2}\frac{-\mathrm{i}}{\not{p}_3 + m}\left(\text{F.T.}\,\langle 0|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)B(x_2)\Psi(x_3)|0\rangle_{\text{amp.}}\right)\xi,
\end{aligned} \tag{8.3}$$

where F.T. stands for the Fourier transform. Compared to the on-shell identity we just changed one scalar into a pseudoscalar. As this is an off-shell identity we need not to distinguish incoming and outgoing particles. The nonvanishing contributions to the contact terms for this SWI are:

$$\text{F.T.}\,\langle 0|\mathrm{T}\bar{\xi}\Psi(x_1)B(x_2)\Psi(x_3)|0\rangle = \frac{-\lambda}{\sqrt{2}}\frac{\mathrm{i}}{p_2^2 - m^2}\frac{-\mathrm{i}}{\not{p}_3 + m}\gamma^5\frac{\mathrm{i}}{\not{p}_1 + \not{k} - m}\xi \tag{8.4a}$$

$$\text{F.T.}\,\langle 0|\mathrm{T}A(x_1)(\mathrm{i}\bar{\xi}\gamma^5\Psi(x_2))\Psi(x_3)|0\rangle = \frac{\lambda}{\sqrt{2}}\frac{\mathrm{i}}{p_1^2 - m^2}\frac{-\mathrm{i}}{\not{p}_3 + m}\frac{\mathrm{i}}{\not{p}_2 + \not{k} - m}\gamma^5\xi \tag{8.4b}$$

$$\begin{aligned}
&\text{F.T.}\,\langle 0|\mathrm{T}A(x_1)B(x_2)(-\mathrm{i})\,(\mathrm{i}\not{\partial}_{x_3} + m)\,B(x_3)\gamma^5\xi|0\rangle = \\
&\qquad \frac{-m\lambda}{\sqrt{2}}\frac{\mathrm{i}}{p_1^2 - m^2}\frac{\mathrm{i}}{p_2^2 - m^2}\frac{\mathrm{i}}{(p_3 + k)^2 - m^2}(-\not{p}_3 - \not{k} + m)\,\gamma^5\xi
\end{aligned} \tag{8.4c}$$

$$\frac{-\mathrm{i}\lambda}{\sqrt{2}}\text{F.T.}\,\langle 0|\mathrm{T}A(x_1)B(x_2)(AB)(x_3)\gamma^5\xi|0\rangle = \frac{-\mathrm{i}\lambda}{\sqrt{2}}\frac{\mathrm{i}}{p_1^2 - m^2}\frac{\mathrm{i}}{p_2^2 - m^2}\gamma^5\xi \tag{8.4d}$$

To evaluate the 4-point function with the current insertion we rewrite the current $\overline{\xi}\mathcal{J}_\mu$ as $\overline{\mathcal{J}_\mu}\xi$, which is identical due to the Majorana properties of the current and the transformation parameter:

$$\begin{aligned}\overline{\xi}\mathcal{J}_\mu = \overline{\xi}\Big\{ &\mathrm{i}\left((\mathrm{i}\partial\!\!\!/ - m)A\right)\gamma_\mu\Psi + \left((\mathrm{i}\partial\!\!\!/ + m)B\right)\gamma^5\gamma_\mu\Psi \\ &- \frac{\mathrm{i}\lambda}{2\sqrt{2}}\gamma_\mu\left(A^2 - B^2\right)\Psi - \frac{\lambda}{\sqrt{2}}\gamma_\mu\gamma^5 AB\Psi\Big\} \\ = \Big\{&\overline{\Psi}\gamma_\mu\mathrm{i}\left(\mathrm{i}\partial\!\!\!/ + m\right)A - \overline{\Psi}\gamma_\mu\left(\mathrm{i}\partial\!\!\!/ + m\right)B\gamma^5 \\ &+ \frac{\mathrm{i}\lambda}{2\sqrt{2}}\overline{\Psi}\gamma_\mu\left(A^2 - B^2\right) - \frac{\lambda}{\sqrt{2}}\overline{\Psi}\gamma_\mu\gamma^5 AB\Big\}\xi \end{aligned} \tag{8.5}$$

This brings the propagator of the (matter) fermion to the left. Again there are four diagrams for the Green function with current insertion:

$$+ \qquad\qquad + \qquad\qquad + \tag{8.6}$$

For the sign of the fermion propagator one has to take care of the momentum flow.

$$\begin{aligned}\mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)B(x_2)\Psi(x_3)\middle|0\right\rangle_{\mathrm{amp.}}\xi &= -\frac{\mathrm{i}\lambda}{\sqrt{2}}\gamma^5\frac{\mathrm{i}}{p\!\!\!/_1 + k\!\!\!/ - m}\gamma_\mu\left(p\!\!\!/_1 + m\right)\xi \\ + \frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{\mathrm{i}}{p\!\!\!/_2 + k\!\!\!/ - m}&\gamma_\mu\left(p\!\!\!/_2 + m\right)\gamma^5\xi - \frac{\lambda}{\sqrt{2}}\gamma_\mu\gamma^5\xi \\ &- \frac{\mathrm{i}m\lambda}{\sqrt{2}}\frac{\mathrm{i}}{(p_3 + k)^2 - m^2}\gamma_\mu\left(p\!\!\!/_3 + k\!\!\!/ - m\right)\gamma^5\xi \end{aligned} \tag{8.7}$$

Dotting the momentum $k_\mu = -(p_1 + p_2 + p_3)_\mu$ into this expression yields

$$\frac{\mathrm{i}}{p\!\!\!/_3 + m}\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}\; k^\mu \mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)B(x_2)\Psi(x_3)\middle|0\right\rangle_{\mathrm{amp.}}\xi$$

$$\begin{aligned} &= \frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{p\!\!\!/_3 + m}\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}\cdot\Big\{\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right) \\ &\quad - \frac{1}{p\!\!\!/_2 + p\!\!\!/_3 - m}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\left(p\!\!\!/_1 - m\right) \\ &\quad - \frac{1}{p\!\!\!/_1 + p\!\!\!/_3 + m}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\left(p\!\!\!/_2 + m\right) \\ &\quad + m\frac{1}{(p_1 + p_2)^2 - m^2}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\left(p\!\!\!/_1 + p\!\!\!/_2 + m\right)\Big\}\gamma^5\xi \\ &= \frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{p\!\!\!/_3 + m}\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\gamma^5\xi \end{aligned}$$

$$
\begin{aligned}
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}(\not{p}_1-m)\,\gamma^5\xi \\
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{p_2^2-m^2}\frac{1}{\not{p}_2+\not{p}_3-m}\gamma^5\xi \\
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}(\not{p}_2+m)\,\gamma^5\xi \\
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{p_1^2-m^2}\frac{1}{\not{p}_1+\not{p}_3+m}\gamma^5\xi \\
&+\frac{\mathrm{i}m\lambda}{\sqrt{2}}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\frac{1}{(p_1+p_2)^2-m^2}(\not{p}_1+\not{p}_2+m)\,\gamma^5\xi \\
&+\frac{\mathrm{i}m\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\gamma^5\xi
\end{aligned}
\tag{8.8}
$$

The third, fifth and sixth term equal the ones from the linearly transformed fields of the r.h.s.:

$$
\begin{aligned}
-\frac{\mathrm{i}\lambda}{\sqrt{2}}\Bigg\{&\frac{1}{p_2^2-m^2}\frac{1}{\not{p}_3+m}\frac{1}{\not{p}_2+\not{p}_3-m}+\frac{1}{p_1^2-m^2}\frac{1}{\not{p}_3+m}\frac{1}{\not{p}_1+\not{p}_3+m} \\
&-m\frac{1}{p_1^2-m^2}\frac{1}{p_2^2-m^2}\frac{1}{(p_1+p_2)^2-m^2}(\not{p}_1+\not{p}_2+m)\Bigg\}\gamma^5\xi
\end{aligned}
\tag{8.9}
$$

The remaining terms add up to:

$$
\begin{aligned}
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\frac{1}{\not{p}_3+m}\Big\{-\not{p}_1-\not{p}_2-\not{p}_3+\not{p}_1-m+\not{p}_2+m-m\Big\}\gamma^5\xi \\
&\qquad=\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\gamma^5\xi
\end{aligned}
\tag{8.10}
$$

This equals the single term coming from the local operator insertion, so that the Ward identity is indeed fulfilled.

8.2 Currents and SWI in the O'Raifeartaigh model

Taking the general formula (**??**) derived in appendix **??** we can derive the supersymmetric current for the O'Raifeartaigh model (short: OR model). From the superpotential in which the superfields have been substituted by their scalar components

$$
f(\phi_1,\phi_2,\phi_3)=\lambda\phi_1+m\phi_2\phi_3+g\phi_1\phi_2^2 \tag{8.11}
$$

we can read off the derivatives with respect to the scalar fields (there is no difference whether we take the mixings of the fields into account first and take the derivatives afterwards or vice versa):

$$
\frac{\partial f(\phi_1,\phi_2,\phi_3)}{\partial\phi_1}=\lambda+g\phi_2^2 \quad=\quad \lambda+\frac{g}{2}\left(A^2-B^2+2\mathrm{i}AB\right) \tag{8.12}
$$

$$
\frac{\partial f(\phi_1,\phi_2,\phi_3)}{\partial\phi_2}=m\phi_3+2g\phi_1\phi_2=\quad m\Phi+\sqrt{2}g\,(A+\mathrm{i}B) \tag{8.13}
$$

$$
\frac{\partial f(\phi_1,\phi_2,\phi_3)}{\partial\phi_3}=m\phi_2 \quad=\quad \frac{m}{\sqrt{2}}\,(A+\mathrm{i}B) \tag{8.14}
$$

After inserting these derivatives and sorting the terms we get

$$\begin{aligned}\mathcal{J}^\mu = & -\sqrt{2}(\partial\!\!\!/\phi)\gamma^\mu\mathcal{P}_R\chi - \sqrt{2}(\partial\!\!\!/\phi^*)\gamma^\mu\mathcal{P}_L\chi - \sqrt{2}\mathrm{i}\lambda\gamma^\mu\chi + \mathrm{i}\mathcal{P}_L\left((\mathrm{i}\partial\!\!\!/ - m)A\right)\gamma^\mu\Psi \\ & + \mathrm{i}\mathcal{P}_R\left((\mathrm{i}\partial\!\!\!/ - m)A\right)\gamma^\mu\Psi^c + \mathcal{P}_L\left((\mathrm{i}\partial\!\!\!/ - m)B\right)\gamma^\mu\Psi - \mathcal{P}_R\left((\mathrm{i}\partial\!\!\!/ - m)B\right)\gamma^\mu\Psi^c \\ & + \mathrm{i}\sqrt{2}\mathcal{P}_R\left((\mathrm{i}\partial\!\!\!/ - m)\Phi\right)\gamma^\mu\Psi + \mathrm{i}\sqrt{2}\mathcal{P}_L\left((\mathrm{i}\partial\!\!\!/ - m)\Phi^*\right)\gamma^\mu\Psi^c \\ & - \frac{\mathrm{i}g}{\sqrt{2}}\left(A^2 - B^2\right)\gamma^\mu\chi - \sqrt{2}gAB\gamma^\mu\gamma^5\chi - 2\mathrm{i}g\gamma^\mu A\phi\mathcal{P}_L\Psi - 2\mathrm{i}g\gamma^\mu A\phi^*\mathcal{P}_R\Psi^c \\ & + 2g\gamma^\mu B\phi\mathcal{P}_L\Psi - 2g\gamma^\mu B\phi^*\mathcal{P}_R\Psi^c\end{aligned} \tag{8.15}$$

Let us start with a rather trivial example, which relates 2- and 3-point functions in lowest order perturbation theory. We consider the SWI

(8.18)

$$\stackrel{!}{=} \quad \text{F.T.}\left\langle 0\middle|\mathrm{T}\left[\Psi(x_2)\left(\overline{\Psi}(x_1)\mathcal{P}_R\xi\right)\right]\middle|0\right\rangle\delta^4(x-x_1)$$

We have only kept those of the contact terms giving nonvanishing contributions. The right hand side will be calculated first; we adopt the convention that all momenta be incoming. The right hand side is

$$\text{RHS}\,(\textbf{??}) = \frac{\mathrm{i}(-k\!\!\!/_2 + m)}{k_2^2 - m^2}\mathcal{P}_R\xi - \frac{\mathrm{i}(k\!\!\!/_1 + m)}{k_1^2 - m^2 - 2\lambda g}\mathcal{P}_R\xi \;+\; \mathcal{O}(g) \tag{8.19}$$

As mentioned earlier, for the calculation of the left hand side care has to be taken about possible higher orders in perturbation theory which may contribute to this SWI. In these diagrams the linear part of the current will be coupled to the external particles via the Goldstino, wherein the coupling constant combined with the parameter for the spontaneous symmetry breaking λ is responsible for the mass splitting between the participating particles A and Ψ. This will prove important – as we will see soon – for constructing the propagators with the correct poles. The pole of the Goldstino at zero mass always cancels out of those diagrams against the momentum influx from the current. Diagrammatically the left hand side looks like ($k = k_1 + k_2$):

$$\text{LHS}\,(\textbf{??}) = \qquad\qquad + \tag{8.20}$$

The analytical expression for the left hand side (**??**) is

$$\begin{aligned}\text{LHS}\,(\textbf{??}) = \; & \mathrm{i}\partial_\mu^x\left\langle 0\middle|\mathrm{T}\left[\left(\overline{\xi}\mathcal{J}^\mu(x)\right)A(x_1)\Psi(x_2)\right]\middle|0\right\rangle_{(0)} \\ & + \mathrm{i}\partial_\mu^x\cdot\mathrm{i}\int d^4y\left\langle 0\middle|\mathrm{T}\left[\left(\overline{\xi}\mathcal{J}^\mu(x)\right)A(x_1)\Psi(x_2)\mathcal{L}_{\text{int}}\right]\middle|0\right\rangle + \ldots \\ = \; & \mathrm{i}\partial_\mu^x\left\langle 0\middle|\mathrm{T}\left[A(x_1)\Psi(x_2)\mathrm{i}\left(\overline{\Psi}(x)\gamma^\mu\left[(\mathrm{i}\partial\!\!\!/ + m)A(x)\right]\mathcal{P}_R\xi\right)\right]\middle|0\right\rangle \\ & - \partial_\mu^x\int d^4y\left\langle 0\middle|\mathrm{T}\left[\left(\sqrt{2}\mathrm{i}\lambda\overline{\xi}\gamma^\mu\chi(x)\right)\left(\sqrt{2}g\overline{\Psi}(y)\mathcal{P}_R\chi(y)\right)A(x_1)\Psi(x_2)\right]\middle|0\right\rangle\end{aligned}$$

$$
\begin{aligned}
&\quad + \text{higher orders} \\
&\overset{\text{FT}}{\to} \frac{\mathrm{i}(-\not{k}_2 + m)}{k_2^2 - m^2}\,(\not{k}_1 + \not{k}_2)\,\frac{\not{k}_1 + m}{k_1^2 - m^2 - 2\lambda g}\mathcal{P}_R\xi \\
&\quad - \frac{\mathrm{i}(-\not{k}_2 + m)}{k_2^2 - m^2}\frac{2\lambda g}{k_1^2 - m^2 - 2\lambda g}\mathcal{P}_R\xi \; + \; \mathcal{O}(g) \\
&= \frac{\mathrm{i}(-\not{k}_2 + m)}{k_2^2 - m^2}\mathcal{P}_R\xi - \frac{\mathrm{i}(\not{k}_1 + m)}{k_1^2 - m^2 - 2\lambda g}\mathcal{P}_R\xi \; + \; \mathcal{O}(g) \quad = \quad \text{RHS}\,(\mathbf{??}) \quad \surd
\end{aligned} \tag{8.21}
$$

The SWI is fulfilled. Amputating the external legs (except for the current) by means of the LSZ reduction formula produces the on-shell identity which is thence automatically fulfilled as well.

Chapter 9

Gauge theories and Supersymmetry

In gauge theories there appears a new phenomenon not met in the previous chapters: the participation of (massless or massive) vector bosons connected to the concept of gauge symmetry and gauge transformations. These are indispensable ingredients for a realistic field theoretic model describing elementary particle phenomenology. The gauge principle, i.e. the covariance of the fields under local phase transformations, must in a supersymmetric field theory be incorporated in a SUSY covariant manner. As shown in [**?**] and [**?**] the kinetic terms with minimal coupling can be written down in a SUSY-covariant form by introducing a vector superfield $\hat{V}$ (this is a real superfield with $\hat{V}^\dagger = \hat{V}$), and making the replacement

$$S_{\text{kin}} = \int d^4x \, \frac{1}{2} \left[\hat{\Phi}^\dagger \hat{\Phi}\right]_D \longrightarrow \int d^4x \, \frac{1}{2} \left[\hat{\Phi}^\dagger e^{\pm c\hat{V}} \Phi\right]_D . \tag{9.1}$$

Therein c is a normalization constant depending on the normalization of the algebra of the gauge symmetry which is as changing from author to author as the choice of sign. The sign of c is related to the sign in the gauge-covariant derivative,

$$D_\mu = \partial_\mu \pm \mathrm{i}g \sum_a T^a A^a_\mu . \tag{9.2}$$

The kinetic term for the gauge fields is produced with the help of spinor superfields, chiral superfields equipped with an additional spinor index. They are established by triply applying the super-covariant derivative $\mathcal{D}$ to the vector superfield

$$\hat{W}(x,\theta) = -\frac{1}{4} \left(\overline{\mathcal{D}}\mathcal{D}\right) \mathcal{D} \, \hat{V}(x,\theta). \tag{9.3}$$

Then the kinetic part of the gauge fields can be expressed as

$$S_{\text{gauge}} = \frac{1}{2} \int d^4x \, \mathrm{Re}\Big[\sum_a \overline{W^a_R} W^a_L\Big]. \tag{9.4}$$

There is a high redundancy in the superfield formulation of supersymmetric gauge theories. The new superfield $\hat{V}$ there contains a huge amount of unphysical degrees of freedom. But we can get rid of them. The kinetic part (and the superpotential as well) are not only invariant under SUSY and gauge transformations but also under so called extended gauge transformations. These are gauge transformations where the gauge parameter (usually a scalar spacetime dependent parameter) is replaced by a complete superfield $\hat{\Lambda}(x,\theta)$. We can use these transformations to gauge away the superfluous degrees of freedom, three scalar and one spinor component field so that only the gauge field, the gaugino and a scalar field with canonical dimension two remain. This is called the Wess-Zumino gauge. After having fixed the above mentioned components, only the ordinary gauge transformations survive from the extended gauge transformations.

The Lagrangean density of the matter fields with minimal coupling therefore has the structure:

$$\begin{aligned}\mathcal{L}_{\text{mat}} = (D_\mu\phi)^\dagger (D^\mu\phi) + \frac{\mathrm{i}}{2}\left(\overline{\Psi}\not{D}\Psi\right) + F^\dagger F - \sqrt{2}g\overline{\lambda^a}\phi^\dagger T^a\Psi_L \\ - \sqrt{2}g\overline{\Psi_L}T^a\phi\lambda^a + g\phi^\dagger T^a\phi D^a + \mathcal{W}(\phi,\Psi,F)\end{aligned} \tag{9.5}$$

Here $\mathcal{W}(\phi,\Psi,F)$ stands for the superpotential parts of the matter Lagrangean density which are globally and locally invariant under the gauge symmetry group. It does not contain any derivatives of the fields.

The kinetic terms of the gauge fields and gauginos are:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{\mathrm{i}}{2}\overline{\lambda^a}\left(\not{D}\lambda\right)^a + \frac{1}{2}D^aD^a \tag{9.6}$$

Since it is consistent with the gauge symmetry, we may add a *Fayet-Iliopoulos* term

$$\mathcal{L}_{\text{FI}} = \zeta^a D^a \qquad \text{with} \qquad f^a_{bc}\zeta^a = 0. \tag{9.7}$$

The last condition is necessary in the non-Abelian case for this term to transform into a total derivative under SUSY. It forces the gauge field part in the covariant derivative of the gauginos produced when SUSY-transforming the auxiliary field to vanish.

9.1 The de Wit–Freedman transformations

The Wess-Zumino supergauge fixing procedure destroys invariance of the Lagrangean density under SUSY transformations as well as under extended gauge transformations. When performing a SUSY transformation the states gauged away in the WZ gauge are populated again with the effect that the Lagrangean density is no longer WZ gauged. This can be remedied by performing another extended gauge transformation to newly reach WZ gauge. From last section's discussion this is understandable from the fact that SUSY and gauge transformations are not completely orthogonal to each other. The several transformations and their relations are displayed below:

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\mathcal{T}_{\text{SUSY}}"] \arrow[d, "\mathcal{T}_{\text{ext.g.}}"'] & \mathcal{L} \arrow[d, "\mathcal{T}_{\text{ext.g.}}"] \\ \mathcal{L}^{WZ} \arrow[r, "\mathcal{T}_{\text{deW-F}}"'] \arrow[ur, "\mathcal{T}_{\text{SUSY}}"] & \mathcal{L}^{WZ} \end{tikzcd}$$

When performing a SUSY transformation and an extended gauge transformation afterwards (for the details cf. [?]) this results in a combined transformation called *de Wit–Freedman transformation* which leaves the Lagrangean density in WZ gauge invariant [?]. In de Wit–Freedman transformations the spacetime derivatives are replaced by gauge covariant derivatives; furthermore there are some additional terms. So de Wit–Freedman transformations are the gauge-covariant version of the SUSY transformations. For supersymmetric Yang–Mills theories they are (we put a tilde on them to distinguish them from the ordinary supersymmetry transformations; for more details see appendix **??**):

$$\boxed{\begin{aligned}\tilde{\delta}_\xi\phi &= \sqrt{2}\left(\bar{\xi}\Psi_L\right),\\ \tilde{\delta}_\xi\psi &= -\,\mathrm{i}\sqrt{2}\gamma^\mu\left((D_\mu\phi)\mathcal{P}_R+(D_\mu\phi)^\dagger\mathcal{P}_L\right)\xi+\sqrt{2}(F\mathcal{P}_L+F^\dagger\mathcal{P}_R)\xi,\\ \tilde{\delta}_\xi F &= -\,\mathrm{i}\sqrt{2}\left(\bar{\xi}\not{D}\Psi_L\right)+2g\bar{\xi}T^a\phi\lambda^a_R,\\ \tilde{\delta}_\xi A^a_\mu &= -\left(\bar{\xi}\gamma_\mu\gamma_5\lambda^a\right),\\ \tilde{\delta}_\xi\lambda^a &= -\frac{\mathrm{i}}{2}F^a_{\mu\nu}\gamma^\mu\gamma^\nu\gamma^5\xi+D^a\xi,\\ \tilde{\delta}_\xi D^a &= -\,\mathrm{i}\bar{\xi}\left(\not{D}\lambda\right)^a.\end{aligned}} \tag{9.8}$$

9.2 The current in supersymmetric Yang–Mills theories

Because it is a complicated and lengthy topic we postpone the detailed derivation of the supersymmetric current for supersymmetric Yang–Mills theories (SYM) to the appendix, **??**. We simply state the result for the SUSY current in a supersymmetric Yang-Mills theory

$$\boxed{\begin{aligned}\mathcal{J}^\mu = &-\sqrt{2}\gamma^\nu\gamma^\mu(D_\nu\phi)^T\Psi_R-\sqrt{2}\gamma^\nu\gamma^\mu(D_\nu\phi)^\dagger\Psi_L-\mathrm{i}\gamma^\mu\zeta^a\lambda^a\\ &+\frac{1}{2}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5F^a_{\alpha\beta}\lambda^a-\mathrm{i}g\gamma^\mu\left(\phi^\dagger\vec{T}\phi\right)\cdot\lambda\\ &-\mathrm{i}\sqrt{2}\gamma^\mu\left(\frac{\partial f(\phi)}{\partial\phi}\right)^T\Psi_L-\mathrm{i}\sqrt{2}\gamma^\mu\left(\frac{\partial f(\phi)}{\partial\phi}\right)^\dagger\Psi_R\end{aligned}}\quad. \tag{9.9}$$

It is conserved,

$$\boxed{\partial_\mu\mathcal{J}^\mu=0}\quad, \tag{9.10}$$

as will also be proven in the appendix, **??**.

9.3 Comparison of the currents – physical interpretation

The use of the de Wit–Freedman transformation is not mandatory [?]. It is also possible to use the "ordinary" SUSY transformations to calculate the current. We do want to show now that the current in SYM theories remains the same when using SUSY